THE EXPONENTIAL MAP AT A CUSP SINGULARITY

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Abstract. We study spaces with a cusp singularity embedded in a smooth Riemannian manifold and analyze the geodesics in these spaces which start at the singularity. This provides a basis for understanding the intrinsic geometry of such spaces near the singularity. We show that these geodesics combine to naturally define an exponential map based at the singularity, but that the behavior of this map can deviate strongly from the behavior of the exponential map based at a smooth point or at a conic singularity: While it is always surjective near the singularity, it may be discontinuous and non-injective on any neighborhood of the singularity. The precise behavior of the exponential map is determined by a function on the link of the singularity which is an invariant – essentially the only boundary invariant – of the induced metric. The results are proved in the more general natural setting of manifolds with boundary carrying a so-called cusp metric.

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1. Introduction

Geodesics are among the fundamental objects of differential geometry. On a smooth Riemannian manifold the geodesics starting at a point p are classified by and smoothly depend on their initial velocity vector, and combine to define the exponential map based at p, which in turn yields normal coordinates and important geometric information about the manifold. Also, geodesics arise in the study of the

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propagation of waves on the manifold as paths along which singularities of solutions of the linear wave equation travel.

Much less is known about the behavior of geodesics on *singular* spaces. Here by a geodesic we always mean a locally shortest curve. Our general aim is to analyze the full local asymptotic behavior of the family of geodesics reaching, or starting at, a singularity. Previously, Bernig and Lytchak [BL] obtained first order information for geodesics on general real algebraic sets $X \subset \mathbb{R}^n$, by showing that any geodesic reaching a singular point p of X in finite time must have a limit direction at p. In the special situation of isolated conic singularities p, Melrose and Wunsch [MW] obtain full asymptotic information by showing that the geodesics starting at p define a smooth foliation of a neighborhood of p and thus may be combined to define a smooth exponential map based at p, cf. Theorem 1.2 below.

In this paper we analyze the family of geodesics starting at an isolated cusp singularity, defined below. Our method is based on the geodesic differential equations and yields full asymptotic information about the geodesics. We will see that there is a much richer range of possible local behavior than in the case of conic singularities; for example, there is a natural notion of exponential map based at the singularity, but this map may be non-injective or discontinuous on any neighborhood of the singularity.

A natural setting for our investigation is the notion of *cusp manifold*. This is a smooth (that is, C^∞) manifold, M, with compact boundary, equipped with a semi-Riemannian metric, $\mathbf{g}$, which is Riemannian in the interior $\mathring{M}$ and in a neighborhood U of the boundary ∂M can be written

$$\mathbf{g}_{|U} = (1 - 2r^2 S + O(r^3))\, dr^2 + r^4 h \tag{1}$$

where r is a boundary defining function for M (that is, $r \in C^\infty(M)$ is positive in $\mathring{M}$, vanishes on ∂M and satisfies $dr_{\mathbf{p}} \neq 0$ for all $\mathbf{p} \in \partial M$), S is a smooth function on ∂M and h is a smooth symmetric two-tensor on U whose restriction to ∂M, denoted $\mathbf{g}_{\partial M}$, is positive definite. We call $\mathbf{g}$ a *cusp metric*. More precisely, this is a cusp metric of order two. Cusp metrics of higher order are defined in Section 8, where we also show how our results carry over to this more general case. A given cusp metric defines invariantly the quantities

$$S \in C^\infty(\partial M) \quad \text{and} \quad \mathbf{g}_{\partial M},\ \text{a Riemannian metric on } \partial M$$

(more precisely, S is only determined up to an additive constant) and fixes r to third order. For simplicity we will fix a boundary defining function r throughout.

Cusp manifolds arise as resolutions of spaces X with isolated cusp singularity $p \in X$ (of order two). These are subsets $X \subset \mathbb{R}^n$ so that $X \setminus p$ is a submanifold of $\mathbb{R}^n$ and so that X is given by the following local model in a neighborhood U' of p: Let $p = 0$. Then, with $\mathbb{R}_+ = [0, \infty)$,

$$X \cap U' = \beta(\widetilde{X}), \quad \beta : \mathbb{R}^{n-1} \times \mathbb{R}_+ \to \mathbb{R}^n, \quad (u, z) \mapsto (z^2 u, z) \tag{2}$$

where $\widetilde{X} \subset \mathbb{R}^{n-1} \times \mathbb{R}_+$ is a smooth manifold with boundary, whose boundary is contained in $E = \mathbb{R}^{n-1} \times \{0\}$ and so that $\widetilde{X}$ intersects E transversally. We call $\widetilde{X}$ the *resolution* of X. See Figure 1. Now if g is any smooth Riemannian metric on the ambient space $\mathbb{R}^n$ then the restriction of g to the manifold $X \setminus p$ pulls back under β to a Riemannian metric on the interior of $\widetilde{X}$, which extends smoothly to the boundary. The extension is a cusp metric on $\widetilde{X}$. In this sense the resolution of a

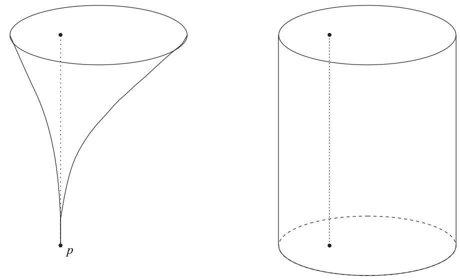

FIGURE 1. A cusp surface singularity X and its resolution $\widetilde{X}$; the dotted lines are explained in Remark 2.4

space with cusp singularity is a cusp manifold. See Section 2 for a discussion of cusp manifolds, cusp singularities and the meaning of S and $\mathbf{g}_{\partial M}$ for cusp singularities.

Let $(M, \mathbf{g})$ be a cusp manifold. Since $\mathbf{g}$ is Riemannian over the interior $\mathring{M}$, locally shortest curves in $\mathring{M}$ are precisely the unit speed solutions of the geodesic differential equations. We call these solutions *geodesics on* M. We are interested in those geodesics *starting at the boundary*, that is, geodesics $\gamma : (T_-, T_+) \to \mathring{M}$ for which $\lim_{\tau\to T_-} r(\gamma(\tau)) = 0$. Here $T_- \in \mathbb{R} \cup \{-\infty\}$. If $\lim_{\tau\to T_-} \gamma(\tau)$ exists then we call it the *starting point* of γ.

In our first theorem we show that, under certain assumptions, geodesics starting at the boundary have a starting point on ∂M, and characterize the possible starting points. Recall that a critical point of $S \in C^\infty(\partial M)$ is a point $\mathbf{p} \in \partial M$ where the differential $dS_{|\mathbf{p}}$ vanishes.

Theorem 1.1. *Let $(M, \mathbf{g})$ be a cusp manifold. Assume that either S is constant or the critical points of S are isolated.*

Then there is $r_0 > 0$ so that the following is true: Let $\gamma : (T_-, T_+) \to \mathring{M}$ be a maximal geodesic and assume that for some $T \in (T_-, T_+)$ the initial part $\gamma_{|(T_-,T)}$ is contained in $\{r < r_0\}$. Then T_- is finite and the starting point $\lim_{\tau\to T_-} \gamma(\tau)$ exists. It is a point on ∂M which is a critical point of S.

Conversely, each critical point of S is the starting point of a geodesic on M.

Our next theorem says that for constant S we have a smooth foliation by geodesics of a neighborhood of p, analogous to the Theorem of Melrose and Wunsch [MW] in the conic case.

Theorem 1.2. *Let $(M, \mathbf{g})$ be a cusp manifold. Assume S is constant. Then to each $\mathbf{p} \in \partial M$ there is a unique geodesic $\gamma_{\mathbf{p}}$ starting at $\mathbf{p}$ at time $\tau = 0$. Furthermore, there is $\tau_0 > 0$ and a neighborhood U of ∂M in M such that the exponential map*

$$\exp_{\partial M} : \partial M \times [0, \tau_0) \to U \subset M, \quad (\mathbf{p}, \tau) \mapsto \gamma_{\mathbf{p}}(\tau)$$

is a diffeomorphism.

See the left picture in Figure 2. For a general cusp manifold, where S need not be constant, it seems clear that a full neighborhood of the boundary is covered by geodesics starting at the boundary, since for any interior point of M any shortest curve from this point to the boundary must be a geodesic. We will not attempt to make this argument precise here (that is, prove existence of a minimizer) but

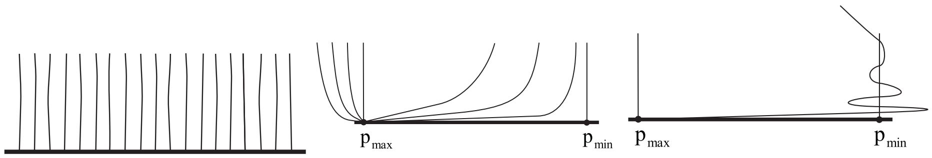

FIGURE 2. Geodesics in the cases $S = \text{const}$, $S_{\varphi\varphi} < 9/4$ and $S_{\varphi\varphi}(\mathbf{p}_{\min}) > 9/4$. The bold line is ∂M. The points $\mathbf{p}_{\max}$ and $\mathbf{p}_{\min}$ are a maximum resp. minimum of S.

rather analyze the case where S is a Morse function much more precisely, mostly in the case of surfaces.

For our next theorems we assume that S is a Morse function. That is, all critical points are non-degenerate (i.e. the Hessian of S at these points is a non-degenerate quadratic form). This implies that the critical points are isolated.

The behavior of geodesics hitting a non-degenerate critical point $\mathbf{p}$ of S can be analyzed very precisely in a neighborhood of $\mathbf{p}$. See for example Proposition 4.2 for the case of local maxima and minima of S. We will now focus on surfaces M, for which we can analyze how these local pictures fit together in a full neighborhood of ∂M. First, we show that there is an exponential map. To simplify the exposition we assume in the following theorems that ∂M is connected.

Theorem 1.3. *Let $(M, \mathbf{g})$ be a cusp surface with connected boundary. Assume that S is a Morse function. There is a parametrization $q \mapsto \gamma_q$, $q \in \mathbf{S}^1$, of the geodesics starting at ∂M at time $\tau = 0$ so that the map*

$$\exp_{\partial M} : \mathbf{S}^1 \times (0, \tau_0) \to U \setminus \partial M, \quad (q, \tau) \mapsto \gamma_q(\tau) \tag{3}$$

is defined and surjective for some $\tau_0 > 0$ and some neighborhood U of ∂M.

The full asymptotic behavior of $\exp_{\partial M}$ as $\tau \to 0$ can be described explicitly in terms of S.

The parametrization $q \mapsto \gamma_q$ is uniquely determined, up to homeomorphisms of $\mathbf{S}^1$, by the requirement that it preserves cyclic ordering in a suitable sense. However, $\exp_{\partial M}$ need not be a diffeomorphism, it may even be discontinuous. See Section 6 for details.

In the next two theorems we give conditions under which more can be said about the exponential map. Here and throughout we denote by S_φ, $S_{\varphi\varphi}$ the first and second derivatives of S with respect to a coordinate φ on ∂M.

Theorem 1.4. *Let $(M, \mathbf{g})$ be a cusp surface with connected boundary. Assume that S is a Morse function, that $S_{\varphi\varphi} < \frac{9}{4}$ on ∂M and that $S_{\varphi\varphi}$ never takes the value 2 at any local minimum, where φ is an arc length parameter on ∂M. Then the exponential map (3) is a homeomorphism for suitable τ_0, U.*

In particular, there is a neighborhood of ∂M such that $U \setminus \partial M$ is foliated by geodesics starting at ∂M.

The exponential map (3) extends to the boundary by letting $\exp_{\partial M}(q, 0)$ be the starting point of γ_q. However, unlike in the case of constant S, the extension $\mathbf{S}^1 \times [0, \tau_0) \to U$ is not continuous in q; this is clear since the image of $\mathbf{S}^1 \times \{0\}$ is a finite set with at least two elements, a maximum and minimum of S. In the setting of Theorem 1.4 the cusp metric defines naturally a compactification

M' of the interior $\mathring{M}$ different from M, and the relation between M and M' is similar to a birational map, see Remark 6.7. The exponential map $\exp_{\partial M}$ extends to a homeomorphism from $\mathbf{S}^1 \times [0, \tau_0)$ to a neighborhood of the boundary of M', and could be proved by our methods to have mildly higher regularity than just continuity.

The value $\frac{9}{4}$ in Theorem 1.4 is optimal in the following sense.

Theorem 1.5. *Let $(M, \mathbf{g})$ be a cusp surface with connected boundary. Assume that S is a Morse function and that $S_{\varphi\varphi} > \frac{9}{4}$ at some minimum of S, for an arc length parameter φ on ∂M. Then the exponential map* (3) *is not injective for any $\tau_0 > 0$.*

That is, in any neighborhood U of ∂M there are points through which at least two geodesics starting at the boundary pass.

Theorems 1.4 and 1.5 are illustrated in Figure 2, middle and right.

We also show that the conditions on S in Theorem (1.4) are satisfied if M arises from a cusp singularity X and if $\partial \widetilde{X}$ is contained in the boundary of a strictly convex subset of $\mathbb{R}^{n-1}$ which contains the origin, and is never doubly tangent to a sphere centered at the origin. See Proposition 5.5.

Theorems 1.2, 1.3, 1.4 and 1.5 may be summarized as follows, in the case of surfaces: There is a well-defined exponential map. For constant S it is a smooth diffeomorphism near the boundary. If S is not too far from constant then it is a homeomorphism, though not on the boundary. If S is far from constant then it may be not injective and also discontinuous.

Main ideas, outline of the proofs. Let $\mathbf{g}^*$ be the metric on the cotangent bundle $T^*\mathring{M}$ dual to $\mathbf{g}$. Consider the energy function (Hamiltonian) $\widetilde{E} = \frac{1}{2}\mathbf{g}^*$ and the associated Hamiltonian vector field $\widetilde{\mathbf{W}}$ on $T^*\mathring{M}$. Then geodesics on $\mathring{M}$ are the projections to $\mathring{M}$ of integral curves of $\widetilde{\mathbf{W}}$. Unit speed geodesics correspond to integral curves on the energy hypersurface $\{\widetilde{E} = \frac{1}{2}\}$. The degeneracy of $\mathbf{g}$ at ∂M implies that $\widetilde{E}$ and hence $\widetilde{\mathbf{W}}$ are undefined over ∂M. To make this explicit, let (r, φ), $\varphi = (\varphi_1, \dots, \varphi_{m-1})$ be local coordinates near a boundary point of M and denote by $\xi, \eta = (\eta_1, \dots, \eta_{m-1})$ the dual coordinates on the fibers of T^*M. Since $\mathbf{g}$ is a positive definite quadratic form in dr and $r^2\, d\varphi$ whose coefficients are smooth functions of r, φ up to $r = 0$, the function $\widetilde{E}$ is a positive definite quadratic form in $\xi, \frac{\eta}{r^2}$ with coefficients smooth up to $r = 0$. Therefore, rescaling

$$\theta = \frac{\eta}{r^3}, \quad E(r, \varphi, \xi, \theta) = \widetilde{E}(r, \varphi, \xi, r^3\theta) \tag{4}$$

yields a function E which is smooth up to the boundary $r = 0$, and simple calculations using the specific form (1) of $\mathbf{g}$ show that the associated Hamiltonian vector field $\mathbf{W}$ (which is $\widetilde{\mathbf{W}}$ written in coordinates r, φ, ξ, θ) is $\frac{1}{r}$ times a vector field $\mathbf{V}$ which is smooth up to the boundary $r = 0$ and also tangent to the boundary.

Clearly, $\mathbf{V}$ and $\mathbf{W}$ have the same integral curves up to time reparametrization, so we need to analyze the integral curves of the rescaled vector field $\mathbf{V}$. It is essential for our analysis that $\mathbf{V}$ is sufficiently non-degenerate at its singular points to make a precise analysis of its integral curves possible. For example, the singular points of $\mathbf{V}$ are hyperbolic if S is a Morse function.

It may seem strange to use the rescaling (4) instead of $\mu = \frac{\eta}{r^2}$, which is more naturally associated to cusp manifolds and is sufficient to make the energy a smooth function. But it turns out that the Hamilton vector field, written in coordinates

r, φ, ξ, μ, needs to be multiplied by r^2 instead of r to yield a smooth vector field, and that the resulting vector field is highly degenerate near its singular points, so a precise analysis via linearization would not be possible.

The rescaling (4) may be given an invariant description as follows. This is the natural setting for our results and methods, but is not strictly necessary to understand most of the paper. Introducing θ corresponds to replacing the vector bundle T^*M by a rescaled cotangent vector bundle which we denote by ${}^3T^*M$, defined as the unique vector bundle over M whose space of sections is the set of smooth one forms α on M satisfying $\alpha(V) = O(r^3)$ for all smooth vector fields V tangent to third order to the boundary, i.e. satisfying $dr(V) = O(r^3)$.[1] In coordinates, this space of sections is spanned by dr, $r^3 d\varphi_1, \ldots, r^3 d\varphi_{m-1}$ over $C^\infty(M)$. Then E is a smooth function on ${}^3T^*M$ and the rescaled geodesic vector field $\mathbf{V}$ is a smooth vector field on ${}^3T^*M$.

We also need to describe the relation of ${}^3T^*M$ to $T^*\partial M$. The rescaled cotangent bundle ${}^3T^*M$ is a manifold with boundary $\partial({}^3T^*M) = {}^3T^*_{\partial M}M$, and π^*r (where $\pi : {}^3T^*M \to M$ is the projection), denoted r for short, is a boundary defining function. The coordinate ξ is invariantly defined on ${}^3T^*_{\partial M}M$, and for any $\xi_0 \in \mathbb{R}$ the affine subbundle $\xi = \xi_0$ of ${}^3T^*_{\partial M}M$ may be naturally identified with $T^*\partial M$. This identification is given in coordinates by $\xi_0\, dr + \theta\, r^3 d\varphi \mapsto \theta d\varphi$, where we write $\theta d\varphi = \sum_{i=1}^{m-1} \theta_i d\varphi_i$. That is, in coordinates the embedding $T^*\partial M \to {}^3T^*_{\partial M}M$ is simply given by $(\varphi, \theta) \mapsto (0, \varphi, \xi_0, \theta)$ where on the left θ plays the role of the fiber coordinate on $T^*\partial M$.

Our aim is to understand the dynamics of $\mathbf{V}$ close to the boundary, so the dynamics of the restriction of $\mathbf{V}$ to the boundary plays an important role in our analysis. Unit speed geodesics leaving the boundary have $\xi = 1$ there, so this restriction may be considered as a vector field on $T^*\partial M$ under the identification just described. We will see that the boundary dynamics is given by a damped Hamiltonian system with Hamiltonian

$$E^\partial = S + \tfrac{1}{2}\mathbf{g}^*_{\partial M} \quad \text{ on } T^*\partial M.$$

More precisely, in local coordinates it is given by the equations

$$\dot\varphi = E^\partial_\theta, \quad \dot\theta = -E^\partial_\varphi - 3\theta. \tag{5}$$

Here $E^\partial_\theta = (\frac{\partial E^\partial}{\partial \theta_1}, \ldots, \frac{\partial E^\partial}{\partial \theta_{m-1}})$ etc. The Hamiltonian E^∂ corresponds to a particle moving on the Riemannian manifold $(\partial M, \mathbf{g}_{\partial M})$ in the potential S.

The theorems now follow from a precise analysis of $\mathbf{V}$. Singular points $\overline{\mathbf{p}} \in {}^3T^*_{\partial M}M$ of $\mathbf{V}$ (that is, points where $\mathbf{V}$ vanishes) correspond to critical points $\mathbf{p} \in \partial M$ of S via projection, and integral curves of $\mathbf{V}$ leaving $\overline{\mathbf{p}}$ correspond to geodesics starting at $\mathbf{p}$. These integral curves foliate the unstable manifold $M^u_{\mathbf{p}} \subset {}^3T^*M$ of $\mathbf{p}$. If S is constant then each $\mathbf{p} \in \partial M$ is a critical point, and $\mathbf{V}$ is transversally hyperbolic with respect to the submanifold of ${}^3T^*M$ formed by the corresponding points $\overline{\mathbf{p}}$, and the unstable manifold theorem yields Theorem 1.2. For general S, the analysis of the exponential map breaks up into three parts:

a) Understand the flow on $M^u_{\mathbf{p}}$ for each critical point $\mathbf{p}$.

[1] For simplicity, we assume a fixed boundary defining function r is chosen, although to define ${}^3T^*M$ it is enough to require the choice of r up to changes of the form $r \mapsto a_1 r + a_2 r^2 + O(r^3)$ where $a_1 > 0$, a_2 are constants.

b) Understand the regularity of $M^u = \bigcup_{\mathbf{p}} M^u_{\mathbf{p}}$, that is, how the various $M^u_{\mathbf{p}}$ fit together.
c) Determine whether M^u projects diffeomorphically to M in a neighborhood of ∂M, under the projection ${}^3T^*M \to M$.

If S is a Morse function then the singular points of $\mathbf{V}$ are isolated, and the linear part of $\mathbf{V}$ is invertible at each singular point; this allows to do a) and answer c) affirmatively (for $M^u_{\mathbf{p}}$) near each $\mathbf{p}$. The main task now is to understand the global boundary dynamics, i.e. the global behavior of the unstable manifolds $\Gamma^u_{\mathbf{p}} = \partial M^u_{\mathbf{p}}$ of $\mathbf{V}$ restricted to the boundary. In the case of surfaces this global analysis can be done. This allows to do a), b) and c) in a full neighborhood of the boundary. Here some technicalities about linearizations enter; the assumption $S_{\varphi\varphi} \neq 2$ at minima in Theorem 1.4 is a non-resonance condition needed to ensure existence of a C^1 linearization. However, we believe that the theorem is true without this assumption. From this we then deduce Theorems 1.1, 1.3, 1.4, and 1.5.

Structure of the paper. In Section 2 we introduce cusp manifolds and show how they arise from cusp singularities. In Section 3 we calculate the rescaled geodesic vector field $\mathbf{V}$ and analyze its singular points, and prove Theorem 1.2. In Section 4 we analyze the local behavior of the flow of $\mathbf{V}$ for a Morse function S, and discuss linearizations. In Section 5 we analyze the boundary dynamics, which we use in Section 6 to prove the remaining theorems, after defining the exponential map. In Section 7 we give some examples and in Section 8 we indicate the adjustments needed for higher order cusps. Here the same theorems hold, with essentially the same proofs, except that $\frac{9}{4}$ has to be replaced by $\frac{(2k-1)^2}{2k(k-1)}$ for a cusp of order k.

Further related work. The exponential map may be used to study differential geometric quantities near a singularity. For example, the description of the asymptotic behavior of the exponential map may be used to deduce the asymptotic behavior of the volume of balls centered at the singularity, as the radius tends to zero. In the case where p is a smooth point, the coefficients in this expansion are related to the curvature at p. In [Gri2] such an expansion was derived for real analytic isolated surface singularities; however, the balls were defined extrinsically, i.e. with distance defined as distance in the ambient space $\mathbb{R}^n$. Also, the bi-Lipschitz geometry of (singular) algebraic subsets of $\mathbb{R}^n$ was investigated extensively in [Mo, Pa, Bi, Va1, Va2]. However, very little is known about the precise intrinsic geometry of these sets, and the present paper provides a first insight into this question.

2. The geometric setting

In this section we first define the notion of a cusp singularity of order two of a subset X of a smooth manifold Z. Since this is a local notion, we may assume $Z = \mathbb{R}^n$. We give two equivalent characterizations. Then we introduce the notion of cusp metric on a manifold with boundary, which captures essential features of the metrics induced on a space with a cusp singularity of order 2 from a smooth ambient metric. A similar discussion applies to cusp singularities of higher order.

First definition of a cusp singularity of order 2. Let X be a subset of $\mathbb{R}^n$. The tangent cone $C(X,p)$ of X at $p \in X$ is the set of limits of oriented secant half-lines through p from points in X:

$C(X,p) := \{r\nu :\ r>0, \nu\in \mathbf{S}^{n-1}, \exists (p_k)_k \in X\setminus\{p\} : p_k\to p \text{ and } \lim_k \frac{p_k-p}{|p_k-p|} = \nu\}.$

Assume X is a subset of $\mathbb{R}^n$ so that $X\setminus\{p\}$ is a submanifold. We say that X has a *cusp singularity of order two* at p if the tangent cone $C(X,p)$ is a single half-line and the singularity can be resolved by a single quadratic blowing-up. That is, after choosing coordinates $(x_1,\dots,x_{n-1},z)$ in $\mathbb{R}^n$ so that $p=0$ and $C(X,0)=\mathbb{R}_{>0}\partial_z$ the space X is given as in (2).

Definition of cusp singularity of order 2 in terms of iterated blowings-up. Equivalently, cusp singularities of order 2 can be characterized by an iteration of standard point blowings-up. We explain this here since it lends itself to natural generalizations and since it allows a simple derivation of the standard form for the metric below.

Recall the notion of (oriented) *blowing-up* of a manifold Z in a point $q\in Z$, which is a geometric way of introducing polar coordinates around q. This is a manifold with boundary, denoted $[Z,q]$, together with a smooth map $\beta_q : [Z,q]\to Z$ (the *blowing-down map*) which maps the boundary $\partial[Z,q]$ to q and is a diffeomorphism from $[Z,q]\setminus\partial[Z,q]$ to $Z\setminus\{q\}$, and which is, locally near $\partial[Z,q]$ resp. q, given by the following model: $Z=\mathbb{R}^n$, $q=0$, and then $[Z,q]=\mathbb{R}_+\times\mathbf{S}^{n-1}$ and $\beta_q(r,\omega)=r\omega$. Thus $\partial[Z,q]=\{0\}\times\mathbf{S}^{n-1}$. See [Mel] or [Gri1] for a more in-depth discussion, in particular of the coordinate invariance of this notion and of its generalization to manifolds with corners. Of this we only need the case where Z is itself a manifold with boundary and $q\in\partial Z$. Then $[Z,q]$ is a manifold with corners; the local model is $Z=\mathbb{R}^{n-1}\times\mathbb{R}_+$, $q=0$, $[Z,q]=\mathbb{R}_+\times\mathbf{S}^{n-1}_+$, where $\mathbf{S}^{n-1}_+=\mathbf{S}^{n-1}\cap(\mathbb{R}^{n-1}\times\mathbb{R}_+)$ is the upper half sphere and β_q is as before. In this case, β_q maps $\{0\}\times\mathbf{S}^{n-1}_+$ to 0 and is a diffeomorphism between the complements of these sets. In either case $\beta_q^{-1}(q)$ is called the *front face* of the blowing-up.

If $X\subset Z$ then the *strict transform* of X under the blowing-up of p is the closure of the pre-image of $X\setminus\{q\}$, so $\beta_q^*X := \overline{\beta_q^{-1}(X\setminus\{q\})}$. A *p-submanifold* of a manifold with boundary Z is a submanifold $X\subset Z$ such that $\partial X\subset\partial Z$ and X hits ∂Z transversally. This extends to manifolds with corners Z; we only need the straightforward case where X intersects the boundary only in the interior of a boundary hypersurface.

Let Z be a manifold. We say that $X\subset Z$ has a *conic singularity* at $q\in X$ if $X\setminus\{q\}$ is a submanifold and the strict transform of X is a p-submanifold of $[Z,q]$. If Z is a manifold with boundary and $q\in\partial Z$ then we require, in addition, that ∂X be contained in the interior of the front face of $[Z,q]$.

A subset $X\subset Z$ has a *cusp singularity of order 2* at $p\in X$ if its strict transform $X'=\beta_p^*(X)\subset Z' := [Z,p]$ intersects $\partial Z'$ in a single point q and has a conic singularity there. Thus, the singularity of X can be resolved by first blowing-up p and then q. The resolution is the strict transform

$$\widetilde{X} = (\beta_p\circ\beta_q)^*X. \tag{6}$$

To see the equivalence of the two definitions of cusp singularity of order 2 recall the definition of projective coordinates on $[\mathbb{R}^n,0]$: Let (x,z) be coordinates on $\mathbb{R}^n$ where $x=(x_1,\dots,x_{n-1})\in\mathbb{R}^{n-1}$. Then *projective coordinates* (y,z), $y=(y_1,\dots,y_{n-1})\in\mathbb{R}^{n-1}$, $z\in\mathbb{R}_+$ can be introduced on the 'upper half' $U=\{(r,\omega)\in\mathbb{R}_+\times S^{n-1} : \omega_n>0\}$ of $[\mathbb{R}^n,0]$. These are uniquely determined by requiring that, in terms of these coordinates, the blowing-down map is $\beta_0(y,z)=(zy,z)$. (The

relation to the r, ω-variables is $y_i = \frac{r}{\omega_n}\omega_i$, $z = r\omega_n$, but this is rarely needed.) There are also projective coordinates covering the remainder of $[\mathbb{R}^n, 0]$, but we do not need them here.

The two definitions of cusp singularity of order 2 of $X \subset Z = \mathbb{R}^n$ at $p = 0$ are related as follows. Let $\beta_0 : [\mathbb{R}^n, 0] \to \mathbb{R}^n$ be the blowing-down map. Points of the front face $\partial[\mathbb{R}^n, 0]$ correspond to directions at 0, so $C(X, 0) = \{r\nu : \nu \in \partial(\beta_0^* X), r > 0\}$. Thus, q in the second definition spans the half line $C(X, 0)$. Assume $C(X, 0) = \mathbb{R}_{>0}\partial_z$. Then we may use the projective coordinates introduced above on $Z' = [\mathbb{R}^n, 0]$ near q, so we identify a subset of Z' by $\mathbb{R}^{n-1} \times \mathbb{R}_+$, and then $q = 0$. Now blow-up q in Z' and denote the blowing-down map β_q. Use projective coordinates again, now on $[Z', q] \supset [\mathbb{R}^{n-1} \times \mathbb{R}_+, 0]$. Call these (u, z), so that $\beta_q(u, z) = (zu, z)$. The fact that, in the second definition, the strict transform of X' hits the front face of $[Z', q]$ in its interior means precisely that, in a neighborhood of the boundary, it is contained in the domain of definition of these coordinates. Then the combined blowing-down map $(\beta_p \circ \beta_q)(u, z) = (z^2 u, z)$ is precisely the quadratic blowing-down map β in (2).

Cusp metrics.

Definition 2.1. *A* cusp metric *on a manifold with boundary M is a smooth semi-Riemannian metric* $\mathbf{g}$ *on M for which near any boundary point there are coordinates $(r, \varphi_1, \dots, \varphi_{m-1})$, with the boundary defined by $r = 0$, in which* $\mathbf{g}$ *takes the form*

$$\mathbf{g} = (1 - 2r^2 S + O(r^3))\, dr^2 + r^4 \left(2\sum_{i=1}^{m-1} b_i \, dr\, d\varphi_i + \sum_{i,j=1}^{m-1} c_{ij}\, d\varphi_i\, d\varphi_j \right) \tag{7}$$

where $S = S(\varphi)$, $b_i = b_i(r, \varphi)$ and $c_{ij} = c_{ij}(r, \varphi)$ are smooth and $\sum_{i,j=1}^{m-1} c_{ij}(0, \varphi)\, d\varphi_i\, d\varphi_j$ is positive definite for each φ, so defines a Riemannian metric on the boundary.

Observe that the form of metric (7) is unaffected by a change of the φ-coordinates. However, this is not true for the coordinate (boundary defining function) r. More precisely, we have:

Lemma 2.2. *A cusp metric on M determines invariantly the function S on ∂M, up to additive constants, as well as a Riemannian metric $\mathbf{g}_{\partial M}$ on ∂M, given by $\mathbf{g}_{\partial M} = \sum_{i,j=1}^{m-1} c_{ij}(0, \varphi)\, d\varphi_i\, d\varphi_j$ in coordinates. The metric also fixes the boundary defining function r up to replacing it by $r + br^3 + O(r^4)$, where b is constant on each connected component of ∂M.*

Proof. Under changes of the φ-coordinates the last term in (7) transforms like a Riemannian metric on ∂M. A change of r-coordinate, i.e. setting $r = g_1(\varphi)\rho + g_2(\varphi)\rho^2 + g_3(\varphi)\rho^3 + O(\rho^4)$ will give the required $(1 + O(\rho^2))\, d\rho^2$ behavior iff $g_1 \equiv 1$, $g_2 \equiv 0$, and then $r = \rho + g_3(\varphi)\rho^3$ yields $dr = (1 + 3\rho^2 g_3)\, d\rho + \rho^3 dg_3$ and therefore

$$(1 - 2r^2 S + O(r^3))\, dr^2 = (1 - 2\rho^2 S + 6\rho^2 g_3 + O(\rho^3))\, d\rho^2 + 2\rho^3 d\rho dg_3 + O(\rho^6) \tag{8}$$

so the mixed term remains $O(\rho^4)$ (as required in (7)) iff g_3 is locally constant, and the $d\varphi_i d\varphi_j$ term is changed only by $O(\rho^6)$, so the metric on ∂M is well-defined. $\square$

The reason for introducing the notion of cusp metric is the following proposition.

Proposition 2.3. *Let $X \subset \mathbb{R}^n$ be a space with a cusp singularity of order 2 at p. Let g be a smooth Riemannian metric on $\mathbb{R}^n$ and g_X its restriction to $X \setminus \{p\}$.*

Then the pullback $g_{\widetilde{X}}$ of g_X to $\widetilde{X}\setminus\partial\widetilde{X}$, where $\widetilde{X}$ is the resolution of X, extends to a cusp metric on $\widetilde{X}$.

Proof. We use the second definition of cusp singularity of order 2, because we can use normal coordinates on the first blowing-up space. They do much of the work for us. We first work on $\mathbb{R}^n$ and its successive blowings-up and then restrict to $\widetilde{X}$. Let $p=0$, $Z'=[\mathbb{R}^n,0]$ and $\widetilde{Z}=[Z',q]$, with blowing-down maps $\beta_p: Z'\to\mathbb{R}^n$, $\beta_q:\widetilde{Z}\to Z'$, $\beta=\beta_p\circ\beta_q$ as in (6). We need to find a suitable r-coordinate on $\widetilde{Z}$ so β^*g has the form (7) near the interior of the front face.

Introduce geodesic polar coordinates for g near 0. This means that we choose a diffeomorphism of $Z'=[\mathbb{R}^n,0]$ with $\mathbb{R}_+\times\mathbf{S}^{n-1}$ so that, with R the coordinate on $\mathbb{R}_+$, the metric has the form

$$\beta_p^*g=(dR)^2+R^2h(R)$$

where $R\mapsto h(R)$ is a smooth family of Riemannian metrics on $\mathbf{S}^{n-1}$. In fact, $h(0)$ is the standard metric on $\mathbf{S}^{n-1}$, but this is inessential for the construction.

Next let $U=(U_1,\dots,U_{n-1})$ be any coordinates, centered at q, on $\mathbf{S}^{n-1}$. Then $h(R)=\sum_{ij}h_{ij}(R,U)dU_i\,dU_j$ with h_{ij} smooth. On $\widetilde{Z}=[Z',q]$ use projective coordinates $R, u_i=\frac{U_i}{R}$; they are valid in a neighborhood of the interior of the front face, so in particular on $\widetilde{X}$ near $\partial\widetilde{X}$.

Then $(\beta_q^*h_{ij})(R,U)=h_{ij}(R,Ru)=h_{ij}(0,0)+O(R)$ and $\beta_q^*dU_i=d(Ru_i)=u_i\,dR+R\,du_i$ and so, with

$$S(u)=\sum_{ij}h_{ij}(0,0)u_iu_j,\quad h'=\sum_{ij}h_{ij}(0,0)du_idu_j, \tag{9}$$

we have

$$\beta_q^*h(R)=S\,(dR)^2+R\,dRdS+R^2h'+\tilde{h} \tag{10}$$

where $\tilde{h}=O(R)(dR)^2+\sum_i O(R^2)\,dRdu_i+\sum_{ij}O(R^3)\,du_idu_j$. Then the main term of $\beta^*g=\beta_q^*\beta_p^*g$ is

$$(1+R^2S)\,dR^2+R^3\,dRdS+R^4\,h'$$

and the $\tilde{h}$ part only contributes $O(R^3)(dR)^2+\sum_i O(R^4)dRdu_i+\sum_{ij}O(R^5)du_idu_j$. By the calculation (8) we see that we can make the R^3 part of the mixed term disappear by defining the coordinate r by the relation $R=r-\frac{1}{2}r^3S$, and this yields (7), with $c_{ij}(r,u)=h_{ij}(R,Ru)$.

Now consider the submanifold $\widetilde{X}\subset\widetilde{Z}$. Since it is transversal to $r=0$, it can be locally parametrized as $u=u(r,\varphi)$ with u_φ non-singular at $r=0$ (here $\varphi=(\varphi_1,\dots,\varphi_{m-1})$). Restricting β^*g to $\widetilde{X}$ therefore yields a cusp metric again, where S is restricted to $\widetilde{X}$ and the $d\varphi_id\varphi_j$ term is obtained by restricting the du_idu_j-term to $\widetilde{X}$. □

Remark 2.4. *The proof shows that for a cusp manifold $\widetilde{X}$ arising as resolution of a space X with cusp singularity p the quantities S and $\mathbf{g}_{\partial\widetilde{X}}$ are given as follows. First assume that the ambient space is $\mathbb{R}^n$ with the standard Euclidean metric. Let ν be the unit vector at p so that the cusp points in direction ν (i.e. the tangent cone of X at p is $\mathbb{R}_{>0}\nu$), and let H be the hyperplane through p perpendicular to*

ν. For small $t > 0$ let $H_t = t\nu + H$ and $X_t = -t\nu + (X \cap H_t) \subset H$. Then $\partial\widetilde{X} = \lim_{t\to 0} t^{-2} X_t \subset H$ in the Gromov-Hausdorff sense, and

$$S(\mathbf{u}) = |\mathbf{u}|^2, \quad \mathbf{g}_{\partial\widetilde{X}} = \text{ restriction of } g_{eucl} \text{ to } \partial\widetilde{X} \tag{11}$$

where g_{eucl} is the Euclidean metric on the subspace $H \subset \mathbb{R}^n$ and $|\mathbf{u}|$ is the Euclidean norm. For example, if X is given as in (2) then $H = \mathbb{R}^{n-1}$ with the standard metric.

In general the ambient metric induces the structure of Euclidean vector space on the front face of the quadratic (or iterated standard) blowing-up of the ambient space, and then S and $\mathbf{g}_{\partial\widetilde{X}}$ are given in the same way by restriction to $\partial\widetilde{X}$.

The dotted lines in Figure 1 indicate the tangent cone of X at p and the line $\mathbf{u} = 0$ in its resolution.

Remark 2.5. *The proof shows that the Proposition holds for the more general class of metrics obtained from any conic metric by blowing-up of a boundary point.*

3. The rescaled geodesic vector field

We now investigate the geodesic vector field for a cusp metric (7). We always work in coordinates (r, φ) in which (7) holds and write the metric in simplified notation as

$$\mathbf{g} = (1 - 2r^2 S + O(r^3))\, dr^2 + r^4 2B\, dr\, d\varphi + r^4 C\, d\varphi^2$$

where $d\varphi = (d\varphi_1, \ldots, d\varphi_{m-1})$, $B = (b_1, \ldots, b_{m-1})$, $C = (c_{ij})_{i,j=1,\ldots,m-1}$ are smooth functions of r, φ and S is a smooth function of φ. All vectors will be treated as column vectors. By assumption $C > 0$ at $r = 0$. Here and in the sequel, $O(r)$ denotes r times a smooth function on ∂M, hence can be differentiated.

Let ξ, η denote the cotangent coordinates dual to r, φ, respectively. In order to calculate the metric dual to $\mathbf{g}$, we need to invert the coefficient matrix of $\mathbf{g}$. We use the general formula (for a scalar a, vector b and symmetric matrix c)

$$\begin{pmatrix} a & b^t \\ b & c \end{pmatrix}^{-1} = d^{-1} \begin{pmatrix} 1 & -b^t c^{-1} \\ -c^{-1} b & c^{-1} d + c^{-1} b b^t c^{-1} \end{pmatrix}, \quad d = a - b^t c^{-1} b$$

which can be verified by direct calculation, with $a = 1 - 2r^2 S + O(r^3)$, $b = r^4 B$, $c = r^4 C$. Then $d = 1 - 2r^2 S + O(r^3)$, hence $d^{-1} = 1 + 2r^2 S + O(r^3)$, and we get

$$\begin{aligned} \mathbf{g}^* &= \left(1 + 2r^2 S + O(r^3)\right) \xi^2 + 2\left(-B^t C^{-1} + O(r^2)\right) \xi\eta + r^{-4}\left(C^{-1} + O(r^4)\right) \eta^2 \\ &= \xi^2 + 2r^2 G(r, \varphi, \xi, \frac{\eta}{r^3}) \end{aligned}$$

($C^{-1}\eta^2$ is to be interpreted as $\eta^t C^{-1} \eta$) where

$$\begin{aligned} G(r, \varphi, \xi, \theta) &= G_0(r, \varphi, \xi, \theta) + \widetilde{G}(r, \varphi, \xi, \theta), \\ G_0 = S(\varphi)\xi^2 + \tfrac{1}{2} C(\varphi)^{-1} \theta^2, &\qquad \widetilde{G} = O(r)\xi^2 + O(r)\xi\theta + O(r^4)\theta^2 \end{aligned} \tag{12}$$

The energy $\widetilde{E}(r, \varphi, \xi, \eta)$ is defined as half the dual metric, so if we rewrite it in the rescaled coordinates (4) we get

$$E(r, \varphi, \xi, \theta) = \frac{1}{2}\xi^2 + r^2 G(\varphi, r, \xi, \theta) \tag{13}$$

The geodesics are projections of integral curves of the Hamiltonian vector field associated with the energy function E and the symplectic form $dr \wedge d\xi + d\varphi \wedge d\eta$,

that is they are the r, φ part of the solutions of the following differential equation:

$$
\begin{aligned}
r' &= \widetilde{E}_\xi \quad \xi' = -\widetilde{E}_r \\
\varphi' &= \widetilde{E}_\eta \quad \eta' = -\widetilde{E}_\varphi
\end{aligned}
\tag{14}
$$

In order to rewrite this using the θ coordinate, we differentiate (4) with respect to the various variables and obtain

$$
\begin{aligned}
E_\xi &= \widetilde{E}_\xi \qquad E_r = \widetilde{E}_r + 3r^2\theta\widetilde{E}_\eta \\
E_\theta &= r^3\widetilde{E}_\eta \quad E_\varphi = \widetilde{E}_\varphi
\end{aligned}
$$

and

$$\theta' = \frac{\eta'}{r^3} - 3\frac{r'}{r}\theta$$

Therefore, in the new coordinates the differential equations (14) are

$$
\begin{aligned}
r' &= E_\xi \qquad \xi' = -E_r + 3\frac{\theta}{r}E_\theta \\
\varphi' &= \frac{1}{r^3}E_\theta \quad \theta' = -\frac{1}{r^3}E_\varphi - 3\frac{\theta}{r}E_\xi
\end{aligned}
\tag{15}
$$

This new system corresponds to a vector field $\mathbf{W}$ that we call *the geodesic vector field*, which is well defined and smooth outside $\{r = 0\}$.
Since $E = \frac{1}{2}\xi^2 + r^2G$ with G smooth as a function of r, φ, ξ, θ by (13), the rescaled geodesic vector field $\mathbf{V} = r\mathbf{W}$ is smooth up to $r = 0$, and its integral curves are the solutions of the system

$$
\begin{aligned}
\dot{r} &= r[\xi + r^2G_\xi] \quad \dot{\xi} = r^2[-2G - rG_r + 3\theta G_\theta] \\
\dot{\varphi} &= G_\theta \qquad\qquad \dot{\theta} = -\left[G_\varphi + 3\xi\theta + 3r^2\theta G_\xi\right]
\end{aligned}
\tag{16}
$$

(Throughout the paper the time variables for integral curves of $\mathbf{W}$, $\mathbf{V}$ are denoted τ, t, respectively, and $\frac{d}{d\tau}$ by a prime and $\frac{d}{dt}$ by a dot.) Unit speed geodesics lie on the energy level $\{E = \frac{1}{2}\}$. This is a $(2m-1)$-dimensional manifold which intersects the boundary $\{r = 0\}$ transversally and whose intersection with the boundary has two connected components corresponding to $\xi = \pm 1$, by (13). In a neighborhood of this intersection each component can be parametrized by (r, φ, θ). The $\dot{r}$ equation shows that $\xi > 0$ for geodesics leaving the boundary. We will always consider this component.

It is important to understand the leading terms of the energy and of the rescaled geodesic vector field near $r = 0$. These live on the energy hypersurface $\{E = \frac{1}{2}\}$, which is non-compact since θ is not restricted to a bounded set (as opposed to η). For the following estimates it is important that they are uniform in all variables in a neighborhood of $\{r = 0\}$. In particular, $O(r)$ means r times a smooth *bounded* function on $\{E = \frac{1}{2}\}$.

Recall (12). Clearly $|S\xi^2 + \widetilde{G}| \le K(\xi^2 + r^2C^{-1}\theta^2)$, and this easily implies that

$$\xi^2 + r^2C^{-1}\theta^2 = 1 + O(r^2) \qquad \text{on } \{E = \tfrac{1}{2}\} \tag{17}$$

In particular ξ and $r\theta$ are bounded on $\{E = \frac{1}{2}\}$ and hence also G_ξ, rG_r are bounded while $\widetilde{G}_\theta = O(r)$, $\widetilde{G}_\varphi = O(r) + O(r\theta)$ (the two error terms are not comparable since

θ is unbounded) and $3\theta G_\theta - 2G = 2C^{-1}\theta^2 + O(1) = r^{-2}\left[2(1-\xi^2) + O(r^2)\right]$ (using (17) again), so (16) yields, with uniform estimates on $\{E = \frac{1}{2}\}$,

$$\begin{aligned} \dot r &= r\xi + O(r^3) & \dot\xi &= 2(1-\xi^2) + O(r^2) \\ \dot\varphi &= (G_0)_\theta + O(r) & \dot\theta &= -(G_0)_\varphi - 3\xi\theta + O(r) + O(r\theta) \end{aligned} \tag{18}$$

Lemma 3.1. *In a neighborhood of $\{r = 0\}$ the singular locus of $\mathbf{V}$ on the energy hypersurface $\{E = \frac{1}{2}\}$ is $\{r = 0, \theta = S_\varphi = 0\}$.*

Proof. By (18) we have $\dot\xi \geq 1 - 2\xi^2$ in a neighborhood of $\{r = 0\}$, so $\dot\xi = 0$ implies $|\xi| \geq \frac{1}{\sqrt{2}}$ and hence $|\dot r| \geq \frac{1}{\sqrt{2}} r + O(r^3)$, so at a singular point of $\mathbf{V}$ we must have $r = 0$ and hence $\xi = \pm 1$. Then $\dot\varphi = C^{-1}\theta = 0$ gives $\theta = 0$, and then $\dot\theta = 0$ gives $S_\varphi = 0$. □

Therefore singularities of $\mathbf{V}$ correspond to critical points of S, that is, to points $\mathbf{p} = \mathbf{p}(\varphi_0) \in \partial M$ where $S_\varphi(\varphi_0) = 0$. We denote the corresponding singular point of $\mathbf{V}$ having $\xi = 1$ and $\theta = 0$ by $\overline{\mathbf{p}} \in \{E = \frac{1}{2}\}$.

Lemma 3.2. *The linear part of $\mathbf{V}$ on the energy hypersurface at a singular point $\overline{\mathbf{p}} = (0, \varphi_0, 1, 0)$ is*

$$\dot r = r, \quad \dot\varphi = \gamma_1 r + C^{-1}\theta, \quad \dot\theta = \gamma_2 r - S_{\varphi\varphi}\cdot(\varphi - \varphi_0) - 3\theta$$

with $S_{\varphi\varphi}, C$ evaluated at $\mathbf{p}$ and for suitable γ_1, γ_2.

The precise values of $\gamma_1, \gamma_2 \in \mathbb{R}^{m-1}$ are inessential for the analysis. $\gamma_1 = -B^t C^{-1}$ and γ_2 is determined by B, C and the $O(r^3)$ term in the coefficient of dr^2 in $\mathbf{g}$. We rewrite the linear part in matrix form:

$$\text{Linear part of } \mathbf{V}\text{:} \quad \begin{pmatrix} \dot r \\ \dot\varphi \\ \dot\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_1 & 0 & C^{-1} \\ \gamma_2 & -S_{\varphi\varphi} & -3\,\mathrm{Id} \end{pmatrix} \begin{pmatrix} r \\ \varphi - \varphi_0 \\ \theta \end{pmatrix} \tag{19}$$

Proof of Theorem 1.2. The proof is analogous to the proof of the result of Melrose & Wunsch in the conic case [MW].
If S is constant then every point on $\mathcal{C} = \{(r, \varphi, \xi, \theta) : r = 0, \theta = 0, \xi = 1\}$ is a singular point of $\mathbf{V}$. Let $\overline{\mathbf{p}} = (0, \varphi_0, 1, 0) \in \mathcal{C}$. The set $\{\varphi = \varphi_0\}$ is transversal to $\mathcal{C}$, and projecting the linear part of $\mathbf{V}$ at $\overline{\mathbf{p}}$ onto the tangent space of this set we obtain

$$\begin{pmatrix} 1 & 0 \\ \gamma_2 & -3\,\mathrm{Id} \end{pmatrix}$$

(delete the second block of rows and columns in the matrix (19)). This has a single eigenvalue 1 and the $(m-1)$-fold eigenvalue -3, so $\mathbf{V}$ is normally hyperbolic along $\mathcal{C}$. The unstable eigenvector v (for the eigenvalue 1) has non-vanishing ∂_r-component. By the stable manifold theorem [PS] there exists a unique smooth unstable submanifold N of dimension m with $\partial N = N \cap \{r = 0\} = \mathcal{C}$, transversal to $\{r = 0\}$, which is the union of trajectories of $\mathbf{V}$ starting at points of $\mathcal{C}$. The restriction of $\mathbf{V}$ to N is a smooth vector field on N which vanishes at $r = 0$, hence $\mathbf{W} = \frac{1}{r}\mathbf{V}_{|N}$ extends smoothly from $N \cap \{r > 0\}$ to all of N by Taylor's theorem. Also, $\mathbf{W}$ has a non-zero ∂_r-component at ∂N by the $\dot r$ equation in (15). Near $r = 0$, N is a graph over (r, φ) since its tangent space is spanned by $T\mathcal{C}$ and v. Therefore, the projection $(r, \varphi, \theta) \mapsto (r, \varphi)$ restricts, near ∂N, to a diffeomorphism $\psi : N \to M$. Therefore, the vector field $\mathbf{W}' = \psi_*(\mathbf{W}|_N)$ is well-defined near ∂M,

and its integral curves are geodesics in M. Since $\mathbf{W}'$ is transversal to ∂M, the result follows from standard facts about flows.

Remark 3.3. *1) This can be generalized to the case when S is constant only on an open subset of ∂M, with the same proof.*
2) By the standard Gauss lemma (which is also applicable here, cf. the footnote in the proof of the conic theorem in [MW]*), the theorem implies that for constant S the normal form* (7) *may be improved: There is a trivialization of a neighborhood U of the boundary, $\partial M \times [0, \tau_0) \cong U$ (given by the flow of the vector field $\mathbf{W}'$ in the proof), so that*

$$\mathbf{g}_{|U} = dr^2 + r^4 h(r)$$

where h is a smooth family of metrics on ∂M.

4. Local behavior of the rescaled geodesic vector field near singularities

Let $\overline{\mathbf{p}}$ be a singular point of the rescaled geodesic vector field $\mathbf{V}$ corresponding to a critical point $\mathbf{p}$ of S. We will determine the local phase portrait of $\mathbf{V}$ near $\overline{\mathbf{p}}$ in the case where $\mathbf{p}$ is a non-degenerate critical point of S.

We first determine the eigenvalues and eigenvectors of the linear part (19) of $\mathbf{V}$ at $\overline{\mathbf{p}}$. Recall that the Hessian of S at a critical point $\mathbf{p}$ is a well-defined symmetric bilinear form on $T_{\mathbf{p}}\partial M$. By way of the metric $\mathbf{g}_{\partial M}$, this form corresponds to a linear endomorphism of $T_{\mathbf{p}}\partial M$, which we call the *Hessian of S relative to* $\mathbf{g}_{\partial M}$ at $\mathbf{p}$. For example, if one chooses coordinates in which $\partial_{\varphi_1}, \dots, \partial_{\varphi_{m-1}}$ are orthonormal with respect to $\mathbf{g}_{\partial M}$ at $\mathbf{p}$ then this is simply the matrix of partial derivatives $S_{\varphi_i\varphi_j}$, and $C = \mathrm{Id}$ at $\mathbf{p}$. A simple calculation using (19) yields:

Lemma 4.1. *The linear part of $\mathbf{V}$ at a singular point $\overline{\mathbf{p}}$ has an eigenvalue $\lambda_1(\mathbf{p}) = 1$ with eigenvector transverse to $r = 0$, and for each eigenvalue a of the Hessian of S relative to $\mathbf{g}_{\partial M}$ at $\mathbf{p}$ and each corresponding eigenvector $u\frac{\partial}{\partial\varphi} = \sum_{i=1}^{m-1} u_i \frac{\partial}{\partial\varphi_i}$ the two eigenvalues*

$$\lambda_2(\mathbf{p}, a) = \frac{-3 + \sqrt{9 - 4a}}{2} \quad \textit{and} \quad \lambda_3(\mathbf{p}, a) = \frac{-3 - \sqrt{9 - 4a}}{2} \tag{20}$$

with eigenvectors

$$\nu_2(\mathbf{p}, a, u) = u\frac{\partial}{\partial\varphi} + \lambda_2(\mathbf{p}, a)(Cu)\frac{\partial}{\partial\theta} \textit{ and } \nu_3(\mathbf{p}, a, u) = u\frac{\partial}{\partial\varphi} + \lambda_3(\mathbf{p}, a)(Cu)\frac{\partial}{\partial\theta} \tag{21}$$

Note that the eigenvalues $\lambda_{2,3}(\varphi, a)$ are non-real if $a > 9/4$.

This allows us to analyze the geodesics near non-degenerate critical points. We only note the cases of maxima and minima of S, other critical points can be treated similarly.

Proposition 4.2. *Let $\mathbf{p}$ be a non-degenerate critical point of S on a cusp manifold M.*

1. *If S has a local maximum at $\mathbf{p}$ then there is a neighborhood U of $\mathbf{p}$ in M so that $U \cap \mathring{M}$ is foliated by geodesics.*
2. *If S has a local minimum at $\mathbf{p}$ then there is a unique geodesic which starts at $\mathbf{p}$.*

Proof. At a local maximum of S, all eigenvalues a of its Hessian are negative, so all eigenvalues of the linear part of $\mathbf{V}$ are real and $\lambda_3(\mathbf{p}, a) < 0 < \lambda_2(\mathbf{p}, a)$. By the unstable manifold theorem, the unstable manifold $M^u_{\mathbf{p}}$ is a smooth m-dimensional submanifold of ${}^3T^*M$ whose tangent space at $\overline{\mathbf{p}}$ is spanned by the eigenspaces corresponding to λ_1 and the various $\lambda_2(\mathbf{p}, a)$. Eigenvectors for λ_1 have non-zero ∂_r-component, and the φ-components of the eigenvectors $\nu_2(\mathbf{p}, a)$ span $T_{\mathbf{p}}\partial M$. Therefore the projection of $M^u_{\mathbf{p}}$ onto M is a diffeomorphism in a neighborhood of $\mathbf{p}$. Since $M^u_{\mathbf{p}} \setminus \overline{\mathbf{p}}$ is locally foliated by trajectories of $\mathbf{V}$, the result for a maximum follows.

At a minimum of S, all values a are positive, so the linear part of V has a unique eigenvalue, $\lambda_1 = 1$, with positive real part. By the unstable manifold theorem, there is a unique trajectory leaving $\overline{\mathbf{p}}$. The result follows. $\square$

From now on we focus on cusp surfaces. In this case there is only one value of a, which we denote by

$$a(\mathbf{p}) = S_{\varphi\varphi}(\mathbf{p})C(\mathbf{p})^{-1}$$

In arc length coordinates this is simply $S_{\varphi\varphi}(\mathbf{p})$. We write $\lambda_{2,3}(\mathbf{p}) = \lambda_{2,3}(\mathbf{p}, a(\mathbf{p}))$ and $\nu_{2,3}(\mathbf{p}) = \nu_{2,3}(\mathbf{p}, a(\mathbf{p}), 1)$.

Remark 4.3. *In the case where $M = \widetilde{X}$ is the resolution of a surface with cusp singularity, the geometric meaning of $a(\mathbf{p})$ is as follows: Recall that $\partial M \subset \mathbb{R}^{n-1}$, where $\mathbb{R}^{n-1}$ is equipped with the standard euclidean scalar product, $S(\mathbf{u}) = |\mathbf{u}|^2$ and $\mathbf{g}_{\partial M}$ is the induced metric. Then*

$$\tfrac{1}{2}a(\mathbf{p}) = 1 + \mathbf{p} \cdot K(\mathbf{p})$$

where $K(\mathbf{p})$ is the curvature vector of ∂M at $\mathbf{p}$. This is easily seen by taking arc length parametrization $\varphi \mapsto \mathbf{u}(\varphi)$, then $S_{\varphi\varphi} = 2(|\mathbf{u}_\varphi|^2 + \mathbf{u} \cdot \mathbf{u}_{\varphi\varphi})$ and $|\mathbf{u}_\varphi| = 1$, $\mathbf{u}_{\varphi\varphi} = K$. Now evaluate at $\mathbf{u} = \mathbf{p}$.

Linearization near a singular point. A triple of pairwise distinct complex numbers $\lambda_1, \lambda_2, \lambda_3$ are *resonant* if there exist $i \in \{1, 2, 3\}$ and $a_1, a_2, a_3 \in \mathbb{N}_0$ with $a_1 + a_2 + a_3 \geq 2$ such that

$$a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 = \lambda_i. \tag{22}$$

This relation, or the triple (a_1, a_2, a_3), is called *a resonance relation of type i* among λ_1, λ_2 and λ_3.

Given a non degenerate local extremum $\mathbf{p}$ of the function S, we observe that we always have the relation $3\lambda_1(\mathbf{p}) + \lambda_2(\mathbf{p}) + \lambda_3(\mathbf{p}) = 0$. Multiplying by any natural number and adding $\lambda_i(\mathbf{p})$ we obtain three families of resonance relations.

We recall the following very useful linearization criterion (Samovol's criterion).

Theorem 4.4 ([Sa])**.** *Let $(\mu_1 \dots \mu_s, \sigma_1, \dots, \sigma_u)$ be the set of eigenvalues of the linear part of a smooth vector field V at $\mathbf{0}$, a singular point of V. Assume that*

$$Re(\mu_s) \leq \ldots \leq Re(\mu_1) < 0 < Re(\sigma_1) \leq \ldots \leq Re(\sigma_u)$$

Assume there exists a positive integer l such that for each resonant relation,

$$\lambda = \sum r_i^- \mu_i + \sum r_j^+ \sigma_j,$$

where λ is an eigen-value, there exists an integer $i \leq s$ or $j \leq u$ such that

$$-lRe(\mu_i) < -Re(r_1^-\mu_1 + \ldots + r_i^-\mu_i) \text{ or } lRe(\sigma_j) < Re(r_1^+\sigma_1 + \ldots + r_i^+\sigma_j).$$

Then the vector fields V is C^l-conjugate to its linear part in a neighborhood of $\mathbf{0}$.

Remark 4.5. *The condition $\lambda_2(\mathbf{p}) \neq -1$ is equivalent to $a(\mathbf{p}) \neq 2$.*

As a consequence of this result and of the remark, we get the following proposition.

Proposition 4.6. *Assume that M is a cusp surface and that $\mathbf{p} \in \partial M$ is a non-degenerate local minimum for the function S for which $a(\mathbf{p}) < \frac{9}{4}$ and $a(\mathbf{p}) \neq 2$. If $\lambda_2(\mathbf{p}) \in \mathbb{Q}$ then $\mathbf{V}$ is C^1 linearizable near $\overline{\mathbf{p}}$, otherwise it is C^2 linearizable.*

Proof. Write $\lambda_i = \lambda_i(\mathbf{p})$. The assumptions imply $\lambda_1 = 1$, $\lambda_3 < -\frac{3}{2} < \lambda_2 < 0$ and $\lambda_2 \neq -1$.

1) Assume λ_2 is irrational. The resonance relations between the eigenvalues are of the form $\delta(3,1,1) + R$, where $R \in \{(1,0,0),(0,1,0),(0,0,1)\}$ and $\delta \in \mathbb{N}$. For each such relation we always find

$$2\lambda_1 < [3\delta + \omega]\lambda_1 \text{ for } \omega = 0 \text{ or } 1.$$

We conclude using Theorem 4.4 that $\mathbf{V}$ is C^2 linearizable at $\overline{\mathbf{p}}$.

2) Assume that $\lambda_2 = -\frac{p}{q} \neq -1$ is a negative rational number written in its irreducible form.

Let (a_1, a_2, a_3) be a resonance relation of the form $a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 = \lambda_1$. Then $a_1 > 1$ since otherwise $0 > a_2\lambda_2 + a_3\lambda_3 = (1 - a_1)\lambda_1 \geq 0$. Hence $a_1\lambda_1 > \lambda_1$.

So for all resonance relations of type 1 we always find $a_1\lambda_1 > \lambda_1$, so that we can take $j = 1, l = 1$ in regards of Samovol's criterion.

Let (a_1, a_2, a_3) be a resonance relation of the form $a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 = \lambda_2$.
If $a_2 \geq 2$ then $a_2(-\lambda_2) > -\lambda_2$.
If $a_2 = 1$ then $a_1 + a_3\lambda_3 = 0$, thus $a_1 \geq 2$ and so $a_1\lambda_1 > \lambda_1$.
If $a_2 = 0$ we check that $(a_1, a_3) \neq (1,1)$ since $\lambda_2 \neq -1$. Thus we either have $a_1\lambda_1 > \lambda_1$ or $a_2(-\lambda_2) + a_3(-\lambda_3) > -\lambda_3$.

So for all resonance relations of type 2 we can take $l = 1$ and, respectively, $i = 1$, $j = 1$, $j = 1$ or $i = 2$, in view of Samovol's criterion.

Let (a_1, a_2, a_3) be a resonance relation of the form $a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 = \lambda_3$.
If $a_2, a_3 \geq 1$ then $a_2(-\lambda_2) + a_3(-\lambda_3) > -\lambda_3$. Note also that the cases $a_2 = a_3 = 0$ and $a_2 = 0, a_3 = 1$ are impossible.
If $a_2 = 1$, $a_3 = 0$ then $\lambda_3 = a_1\lambda_1 + \lambda_2 \geq \lambda_2$, which is impossible by hypothesis.

So for all resonance relations of type 3 we can take $i = 2, l = 1$, in view of Samovol's criterion.

We conclude using Theorem 4.4 that $\mathbf{V}$ is C^1 linearizable at $\overline{\mathbf{p}}$. □

5. Global boundary dynamics

The rescaled geodesic vector field $\mathbf{V}$ is tangent to the boundary $r = 0$ of $\{E = \frac{1}{2}\} \subset {}^3T^*M$, which is the union of two components corresponding to $\xi = \pm 1$. We will study the dynamics of the flow of $\mathbf{V}$ on the part $\xi = 1$. As explained in the introduction this can be identified with the space $T^*\partial M$ with variables φ, θ in local coordinates.

Let

$$E^\partial(\varphi,\theta) = S(\varphi) + \tfrac{1}{2}C^{-1}\theta^2 = (G_0)_{|\xi=1}.$$

By (18) the trajectories of the vector field $\mathbf{V}$ on $T^*\partial M$ are the solutions of the system

$$\dot\varphi = E^\partial_\theta, \qquad \dot\theta = -E^\partial_\varphi - 3\theta \tag{23}$$

Explicitly,

$$\dot\varphi = C^{-1}\theta, \qquad \dot\theta = -S_\varphi(\varphi) - 3\theta - \tfrac{1}{2}(C^{-1})_\varphi\theta^2.$$

The system (23) is a Hamiltonian system modified by the damping term -3θ. It describes motion of a particle on ∂M, with Riemannian metric given by $\mathbf{g}_{\partial M}$, in the potential S and in the presence of damping (for example, friction). The following properties are standard for such systems. We provide a proof for completeness. In this discussion the dimension of M is arbitrary.

Proposition 5.1. *Let M be a cusp manifold. The singular points of $\mathbf{V}$ on $T^*\partial M$ are equal to the critical points of the boundary energy E^∂ and given by the critical points of S:*

$$\{(\varphi,\theta) : \theta = 0,\ S_\varphi(\varphi) = 0\}.$$

Every maximal trajectory γ of $\mathbf{V}$ on $T^*\partial M$ is defined for all times. If S is constant or has only isolated critical points then we have in addition:

a) *As $t\to\infty$, the trajectory $\gamma(t)$ converges to a singular point of $\mathbf{V}$.*
b) *As $t\to-\infty$, either E^∂ is bounded along γ and $\gamma(t)$ converges to a singular point of $\mathbf{V}$, or else $E^\partial(\gamma(t))\to\infty$.*

Proof. Let $t\mapsto\gamma(t) = (\varphi(t),\theta(t))$, $t\in(T_-,T_+)$ be a maximal trajectory of $\mathbf{V}$. From (23) we get

$$\dot E^\partial = -3|\theta|^2 \tag{24}$$

where $|\theta|^2 := C^{-1}\theta^2$. In particular, E^∂ is decreasing along γ. Since E^∂ is bounded below, it must approach a limit as $t\to T_+$. In particular, γ is contained in a compact subset of $T^*\partial M$, which implies $T_+ = \infty$. Fix $T\in(T_-,\infty)$. Equation (24) implies $\int_T^\infty |\theta(t)|^2\,dt < \infty$, and then the boundedness of $\dot\theta$ easily implies that $\theta(t)\to 0$ as $t\to\infty$. Let $\alpha(t) = S_\varphi(\varphi(t))$. The equation for $\dot\theta$ and $\theta\to 0$ imply that $\int_t^{t+1}\alpha(s)\,ds\to 0$ as $t\to\infty$, and boundedness of $\dot\alpha$ again implies that $\alpha(t)\to 0$, so γ converges to the set of critical points of E^∂. If this set is discrete then a) follows immediately. If S is constant then $(C^{-1})^{\cdot} = \dot\varphi(C^{-1})_\varphi = O(\theta)$, hence

$$(C^{-1}\theta)^{\cdot} + 3C^{-1}\theta = O(|\theta|^2) + C^{-1}\dot\theta + 3C^{-1}\theta = O(|\theta|^2) - \tfrac{1}{2}C^{-1}(C^{-1})_\varphi\theta^2 = O(|\theta|^2)$$

which is integrable over $[T,\infty)$. Now $\theta\to 0$ implies that $\lim_{T'\to\infty}\int_T^{T'}(C^{-1}\theta)^{\cdot}\,dt$ exists, hence so does $\lim_{T'\to\infty}\int_T^{T'}3C^{-1}\theta\,dt$ and hence $\lim_{t\to\infty}\varphi(t)$ exists. Choosing T sufficiently big one may assume that all these curves lie in a single coordinate patch. This proves a).

Next, (24) implies $\dot E^\partial = -6E^\partial + O(1)$ which implies $|E^\partial(\gamma(t))| \le Ke^{6|t|}$ for some constant K. Then the same estimate must hold for $|\theta(t)|^2$, and this implies $T_- = -\infty$. Now by monotonicity either $E^\partial(\gamma(t))\to\infty$ as $t\to-\infty$, or else it is bounded. In the latter case (24) implies that $\int_{-\infty}^\infty|\theta(t)|^2\,dt<\infty$. Then the same arguments as in the proof of a) yield b). ∎

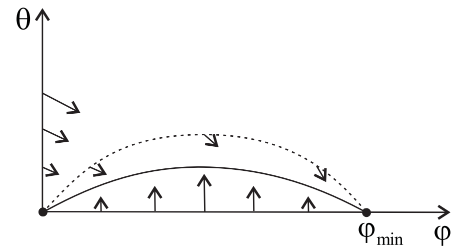

FIGURE 3. Boundary dynamics between a maximum of S (at $\varphi = 0$) and a minimum in case $T = \infty$: The dotted line is the graph of the barrier function f, the solid line is the trajectory $\Gamma_{\mathbf{p}_-}$

Remark 5.2. *The proof carries over almost literally to the following perturbed version of b): Let γ be a solution of the system*

$$\dot{\varphi} = E^{\partial}_{\theta} + f(t), \quad \dot{\theta} = -E^{\partial}_{\varphi} - 3\theta - g(t)$$

where $|f(t)| + |g(t)| \leq Le^{-|t|/K}$ for all $t < 0$, for some constants K, L. Then, if E^{∂} is bounded along γ, then $\gamma(t)$ converges to a singular point of $\mathbf{V}$ as $t \to -\infty$.

We now analyze the case of surfaces M in greater detail. Then we may and will, for simplicity, assume:

$$\varphi \text{ is an arc length parameter for } \partial M. \tag{25}$$

This means $C \equiv 1$ on ∂M, so $\mathbf{V} = \theta\partial_{\varphi} - (S_{\varphi} + 3\theta)\partial_{\theta}$ when restricted to ∂M.

For the following discussion we assume:

$$\begin{gathered} S \text{ has a maximum at } \varphi = 0 \text{ with } S_{\varphi\varphi}(0) < 0 \\ \text{and a minimum at } \varphi_{\min} > 0, \text{ and } S_{\varphi} < 0 \text{ on } (0, \varphi_{\min}). \end{gathered} \tag{26}$$

Denote by $\mathbf{p}_-, \mathbf{p}_{\min}$ the points in ∂M given by $\varphi = 0$, $\varphi = \varphi_{\min}$, respectively. Recall that these correspond to singular points $\overline{\mathbf{p}}_-, \overline{\mathbf{p}}_{\min}$ of $\mathbf{V}$ in $T^*\partial M$, given by these values of φ, and $\theta = 0$. Recall that at a local maximum of S, the linear part of $\mathbf{V}$ restricted to $T^*\partial M$ has eigenvalues $\lambda_3 < 0 < \lambda_2$, with eigenspace for λ_2 spanned by $\nu_2 = \partial_{\varphi} + \lambda_2\partial_{\theta}$. Thus, by the unstable manifold theorem there are (up to time shift) precisely two trajectories starting at $\overline{\mathbf{p}}_-$ at $t = -\infty$, and their directions at $\overline{\mathbf{p}}_-$ are $\pm\nu_2$. We discuss only the trajectory starting in direction ν_2 and denote it by $\Gamma_{\mathbf{p}_-}$, there is an analogous discussion for the other trajectory.

The trajectory $t \mapsto \Gamma_{\mathbf{p}_-}(t) = (\varphi(t), \theta(t))$ starts out at $\overline{\mathbf{p}}_-$ into the first quadrant $\varphi > 0$, $\theta > 0$. It is defined for all times by Proposition 5.1 a). As long as $\varphi < \varphi_{\min}$, that is, for $t < T := \sup\{t : \varphi(s) < \varphi_{\min} \text{ for all } s \leq t\}$, it will stay in the first quadrant since $\mathbf{V}$ points inside this quadrant along this part of its boundary, more precisely:

$$\begin{aligned} \mathbf{V}(\varphi, 0) &= -S_{\varphi}(\varphi)\partial_{\theta} \quad \text{and } -S_{\varphi}(\varphi) > 0 \text{ for } 0 < \varphi < \varphi_{\min} \\ \mathbf{V}(0, \theta) &= \theta\partial_{\varphi} - 3\theta\partial_{\theta} \quad \text{and } \theta > 0 \text{ for the first quadrant.} \end{aligned}$$

See Figure 3. Since $\dot{\varphi} = \theta > 0$ in the first quadrant, φ is strictly increasing on

$(-\infty, T)$. Now there are two possibilities:

$$\text{(27)} \qquad \begin{aligned} &\text{either } T < \infty, \text{ then } \varphi(T) = \varphi_{\min} \text{ and } \theta(T) > 0 \\ &\quad \text{or } T = \infty, \text{ then } \varphi(t) < \varphi_{\min}\ \forall t \text{ and } \varphi(t) \to \varphi_{\min}, \theta(t) \to 0 \text{ as } t \to \infty \end{aligned}$$

(and hence $\Gamma_{\mathbf{p}_-}(t) \to \overline{\mathbf{p}}_{\min}$), by Proposition 5.1 a) since there are no critical points with $\varphi \in (0, \varphi_{\min})$.

We discuss the first case below in the proof of Theorem 1.5. In the second case in (27) the trajectory $\Gamma_{\mathbf{p}_-}$ is the graph of a smooth function $h : (0, \varphi_{\min}) \to (0, \infty)$. We now give a sufficient condition for this to occur. It is obtained by a simple barrier argument.

Lemma 5.3. *Suppose we are in the setup described above, starting from* (25). *Assume that there is a C^1 function $f : [0, \varphi_{min}] \to \mathbb{R}_+$ satisfying the conditions*

$$f(\varphi_{min}) = 0, \text{ and } f(\varphi) > 0 \text{ for } \varphi \in (0, \varphi_{min}) \tag{28}$$

$$f(0) > 0, \text{ or } f(0) = 0 \text{ and } f'(0) > \lambda_2(\mathbf{p}_-). \tag{29}$$

$$ff' + 3f + S_\varphi \geq 0 \text{ on } (0, \varphi_{min}) \tag{30}$$

Then:

a) *The closure of the image of $\Gamma_{\mathbf{p}_-}$ is the graph of a C^1 function $h : [0, \varphi_{min}] \to \mathbb{R}_+$, and $h'(0) = \lambda_2(\mathbf{p}_-)$.*
b) *If in addition $(ff' + 3f + S_\varphi)' < 0$ at φ_{min} then $\Gamma_{\mathbf{p}_-}$ approaches $\mathbf{p}_{min}$ tangent to the eigenvector $\nu_2(\mathbf{p}_{min})$, that is, $h'(\varphi_{min}) = \lambda_2(\mathbf{p}_{min})$.*

Concerning the condition in b), observe that $f(\varphi_{\min}) = 0$ implies differentiability of ff' there, with derivative $(f'(\varphi_{\min}))^2$. Also, the function $g = ff' + 3f + S_\varphi$ satisfies $g(\varphi_{\min}) = 0$ and $g(\varphi) \geq 0$ for $\varphi < \varphi_{\min}$, which implies that $g'(\varphi_{\min}) \leq 0$. The given condition strengthens this to $g'(\varphi_{\min}) < 0$.

Proof. Let
$$D(f) = \{(\varphi, \theta) : 0 \leq \varphi \leq \varphi_{\min},\ 0 \leq \theta \leq f(\varphi)\}$$
We claim that the vector field $\mathbf{V} = \theta\partial_\varphi - (S_\varphi + 3\theta)\partial_\theta$ points inside $D(f)$, or is tangential to the boundary, at each boundary point of $D(f)$: This was checked before the proposition for the parts of the boundary where $\theta = 0$ or $\varphi = 0$. The remaining part of the boundary is the graph of f, which is the zero set of $\tilde{f}(\varphi, \theta) = f(\varphi) - \theta$, with $\tilde{f} > 0$ inside $D(f)$. Now $\mathbf{V}\tilde{f} = \theta f' + S + 3\theta$, which at $\theta = f(\varphi)$ equals $ff' + 3f + S$, so (30) gives $\mathbf{V}\tilde{f} \geq 0$ which was to be shown.

By (29) the trajectory $\Gamma_{\mathbf{p}_-}$ starts out into the interior of $D(f)$, and by what we just proved it can never leave it. Since $\dot{\varphi} = \theta$ is positive in the interior of $D(f)$, the trajectory traces out the graph of a smooth function $\varphi \mapsto h(\varphi)$ on $(0, \varphi_{\min})$. h extends to be smooth at 0 by the unstable manifold theorem, and it is C^1 at $\varphi_{\min}$ since the trajectory is tangent to an eigenspace at $\overline{\mathbf{p}}_{\min}$. This proves a).

Finally, $(ff' + 3f + S_\varphi)' = (f')^2 + 3f' + S_{\varphi\varphi}$ at $\varphi_{\min}$, and if this is negative then f' (everything evaluated at $\varphi_{\min}$) must lie strictly between the zeroes λ_2 and λ_3 of the characteristic polynomial $\lambda^2 + 3\lambda + S_{\varphi\varphi}$ of the linear part of $\mathbf{V}$ at $\varphi_{\min}$, so $h' \geq f' > \lambda_3$. Since h' must be either λ_2 or λ_3 by the local behavior of trajectories, it follows that $h' = \lambda_2$ and hence b). ☐

Proposition 5.4. *Let M be a cusp surface and assume S is a Morse function on ∂M satisfying the bound $S_{\varphi\varphi} < 9/4$ everywhere, where φ is an arc length parameter on ∂M. Let $\Gamma = \bigcup_{\mathbf{p}} \Gamma_{\mathbf{p}}$ where $\mathbf{p}$ ranges over the critical points of S and $\Gamma_{\mathbf{p}}$ is the*

unstable manifold of the point $\overline{\mathbf{p}}$ for the vector field $\mathbf{V}$ on $T^*\partial M$. Then Γ is the graph of a C^1 section of $T^*\partial M$. That is,

$$\Gamma = \{(\varphi,\theta) : \varphi \in \partial M,\ \theta = h(\varphi)\}, \quad h \in C^1(\partial M).$$

Furthermore, $h'(\varphi) = \lambda_2(\mathbf{p}(\varphi))$ at each critical point $\mathbf{p} = \mathbf{p}(\varphi)$ of S.

In fact, the proof shows that h is C^∞ except at minima of S, and at the minima the regularity can, in general, be slightly improved, see the remark after Proposition 6.4.

Proof. Since S is a Morse function and $\dim \partial M = 1$, its critical points are either maxima or minima, and these alternate along ∂M. We first consider an interval between a maximum $\mathbf{p}_-$ and the next minimum $\mathbf{p}_{\min}$. We may assume that the maximum is at $\varphi = 0$ and the minimum is at $\varphi_{\min} > 0$. Then we are in the situation of Lemma 5.3. As barrier function we take $f = -\frac{2}{3}S_\varphi$. This clearly satisfies condition (28). Also, condition (30) holds since

$$g := ff' + 3f + S_\varphi = \tfrac{4}{9}S_\varphi S_{\varphi\varphi} - 2S_\varphi + S_\varphi = S_\varphi(\tfrac{4}{9}S_{\varphi\varphi} - 1) > 0$$

on $(0,\varphi_{\min})$, and $g'(\varphi_{\min}) < 0$ since $S_\varphi = 0$, $\frac{4}{9}S_{\varphi\varphi} - 1 < 0$, $S_{\varphi\varphi} > 0$ at $\varphi_{\min}$. Finally, since $f(0) = 0$ condition (29) is equivalent to $-\frac{2}{3}a > \frac{-3+\sqrt{9-4a}}{2}$ where $a = S_{\varphi\varphi}(0)$, which is easily checked to be true for any $a < 0$. So Lemma 5.3 a) and b) are applicable and yield a function $h_{\mathbf{p}_-\mathbf{p}_{\min}}$ on the interval $[0,\varphi_{\min}]$ satisfying the claim of the Proposition there. We construct corresponding functions for each pair $(\mathbf{p}_-,\mathbf{p}_{\min})$ of consecutive maxima and minima of S, and by changing the signs also for pairs $(\mathbf{p}_{\min},\mathbf{p}_+)$ of consecutive minima and maxima; here $h_{\mathbf{p}_{\min}\mathbf{p}_+}$ is negative in the interior of this interval. All these functions fit together to form a continuous function h on all of ∂M since they vanish at the endpoints of these intervals. h is smooth at maxima $\mathbf{p}$ of S since its graph is the unstable manifold of $\overline{\mathbf{p}}$ near $\overline{\mathbf{p}}$, which is smooth. h is C^1 at minima $\mathbf{p}_{\min}$ of S since its derivatives from both sides agree, by Lemma 5.3b). ☐

The value $\frac{9}{4}$ is optimal for this type of barrier function. Note that the condition $0 < S_{\varphi\varphi} < \frac{9}{4}$ at a minimum $\mathbf{p}_{\min}$ is equivalent to the linear part of $\mathbf{V}$ at $\mathbf{p}_{\min}$ having two distinct negative real eigenvalues.

We now consider the case where $M = \partial\widetilde{X}$ is the resolution of a cusp singularity. Recall from Remark 2.4 that $\partial\widetilde{X}$ is naturally the subset of a Euclidean vector space. If X is given as in (2) then this is simply $\mathbb{R}^{n-1} \times \{0\}$ with the standard metric. In general we still may identify it with $\mathbb{R}^{n-1}$ with the standard metric.

Proposition 5.5. *Let X be a surface with cusp singularity. Assume that $\partial\widetilde{X} \subset \mathbb{R}^{n-1}$ is contained in the boundary of a strictly convex set which contains the origin. Also, assume that, for any point $\mathbf{p} \in \partial\widetilde{X}$ where $\partial\widetilde{X}$ is tangent to the sphere through $\mathbf{p}$ centered at the origin, it is only simply tangent to that sphere. Then we have $S_{\varphi\varphi} < 2$ everywhere on $\partial\widetilde{X}$, where φ is an arc length parameter. In particular, Proposition 5.4 and Theorem 1.4 apply to $M = \partial\widetilde{X}$.*

In the case of a surface in $\mathbb{R}^3$ the simple tangency condition is equivalent to the condition that the osculating circle of $\partial\widetilde{X}$, wherever it is defined, be never centered at the origin. Strict convexity is meant in the sense of nonzero curvature.

Proof. Choose an arc length parametrization $\varphi \mapsto \mathbf{u}(\varphi)$ of $\partial\widetilde{X}$. Then $S(\varphi) = |\mathbf{u}(\varphi)|^2$, so the simple tangency condition is equivalent to S being a Morse function. By Remark 4.3 we have $\frac{1}{2}S_{\varphi\varphi}(\varphi) = 1 + \mathbf{u}\cdot K(\mathbf{u})$ for all $\mathbf{u} = \mathbf{u}(\varphi)$ where $K(\mathbf{u})$ is the curvature vector of $\partial\widetilde{X}$ at $\mathbf{u}$. Suppose $\partial\widetilde{X}$ is contained in the boundary of the strictly convex set $\mathcal{K}$, and let $P_{\mathbf{u}}$ be a supporting hyperplane for $\mathcal{K}$ through $\mathbf{u}$. Then $K(\mathbf{u})$ points into that closed half-space determined by $P_{\mathbf{u}}$ which contains $\mathcal{K}$. Since $\mathcal{K}$ contains the origin, it follows that $\mathbf{u}\cdot K(\mathbf{u}) < 0$, so $S_{\varphi\varphi}(\varphi) < 2 < \frac{9}{4}$ for all φ. □

The proof shows that the conclusion also holds with a certain amount of non-convexity.

6. The exponential map

In this section we prove Theorem 1.1, define the exponential map and then prove Theorems 1.3, 1.4 and 1.5.

Proof of Theorem 1.1. The geodesics are, up to reparametrization, the projections to M of the integral curves of the vector field $\mathbf{V}$ on the energy hypersurface $\{E = \frac{1}{2}\} \subset {}^3T^*M$. Thus, let $\gamma : (T'_-, T'_+) \to {}^3T^*M$ be a maximal integral curve of $\mathbf{V}$, and assume that $\gamma_{|(T'_-,T')}$ is contained in $\{r < r_0\}$ for some $T' \in (T'_-, T'_+)$. The number r_0 will be chosen sufficiently small below. Then necessarily $T'_- = -\infty$ since $\mathbf{V}$ is tangent to the boundary. We will show that $\lim_{t\to-\infty}\gamma(t)$ exists and is a singular point of $\mathbf{V}$.

Write $\gamma(t) = (r(t), \varphi(t), \xi(t), \theta(t))$. Here we have chosen a trivialization of M near the boundary, which allows us to identify ${}^3T^*M$ with $T^*\mathbb{R}_+ \times T^*\partial M$ there, and then (r, ξ) denotes a point in the first factor and (φ, θ) a point in the second factor. The proof falls into two parts: First, we use the $\dot r$ and $\dot\xi$ equations in (18) to show that $\xi(t)$ must be close to one on $(-\infty, T')$, hence $r(t)$ exponentially decreasing as t decreases, and that $\theta(t)$ is bounded. Then the decay of r together with the boundedness of θ shows that the $\dot\varphi$, $\dot\theta$ dynamics is close to the boundary dynamics, which allows us to use Proposition 5.1b).

By (18) there is $K > 0$ such that $\dot\xi \geq 2(1-\xi^2) - Kr^2$ on the energy hypersurface.

Claim 1: $\xi > \frac{1}{2}$ for $t < T'$.

Proof. For r_0 sufficiently small we have $\dot\xi > 1$ for $|\xi| \leq \frac{1}{2}$, so if $\xi(t_0) \leq \frac{1}{2}$ for some $t_0 < T'$ then $\xi(t) < -\frac{1}{2}$ for all $t < t_0 - 1$, and then $\dot r = \xi r + O(r^3)$ implies that r is unbounded as $t \to -\infty$, contradicting the assumption that $r(t) < r_0$ for all $t < T'$. □

Claim 2: $\xi^2 > 1 - Kr^2$ for $t < T'$.

Proof. Let $f(r, \xi) = \xi^2 + Kr^2$. The part of $\gamma_{|(-\infty,T')}$ where $f > 1$ is forward invariant under the flow of $\mathbf{V}$ since at any point with $f(r, \xi) = 1$ we have $2(1-\xi^2) - Kr^2 = Kr^2$ and hence $\frac{1}{2}\mathbf{V}f = \xi\dot\xi + Kr\dot r \geq \xi Kr^2 + Kr(r\xi + O(r^3)) \geq Kr^2 + O(r^4) > 0$, by Claim 1, for r_0 sufficiently small. Hence if $f \leq 1$ at some point $\gamma(t_0)$ then this remains true for all $t \leq t_0$. Now $f(r, \xi) \leq 1$ implies $2(1 - \xi^2) - Kr^2 \geq 1 - \xi^2 > 0$, so we would get $\dot\xi > 1 - \xi^2 > 0$ for all $t \leq t_0$. Then ξ would be monotone increasing in $t \leq t_0$, so $\lim_{t\to-\infty}\xi(t)$ would exist and lie in $[\frac{1}{2}, 1)$. However, this is

impossible since it would imply $\xi(t_0) - \xi(-\infty) > \int_{-\infty}^{t_0}(1-\xi(t)^2)dt = \infty$ and hence unboundedness of ξ. $\square$

From Claim 1 and $\dot{r} = r\xi + O(r^3)$ we obtain $r(t) \leq Ke^{-|t|/3}$, and Claim 2 gives boundedness of $\theta(t)$, using (17). Also, (17) implies that $|\xi(t) - 1| \leq K'e^{-2|t|/3}$. Therefore Remark 5.2, applied to the $\dot{\varphi}$ and $\dot{\theta}$ equations in (18), implies that $\gamma(t)$ approaches a singular point of $\mathbf{V}$ as $t \to -\infty$.

The claim that each singular point $\overline{\mathbf{p}}$ of $\mathbf{V}$ is the starting point of a trajectory of $\mathbf{V}$ follows from the invariant manifold theorem and Lemma 4.1 since the unstable tangent space at $\overline{\mathbf{p}}$ contains the eigenvector for the eigenvalue $\lambda = 1$, which points inside M.

Finally, returning to the unrescaled geodesic vector field we have $r' = \xi + O(r^2)$ along a geodesic, and since ξ is near one this implies that the geodesic reaches $r = 0$ in finite backward time, so T_- is finite. This completes the proof of Theorem 1.1.

Definition of the exponential map. We now define the exponential map in the case where M is a surface with connected boundary and S is a Morse function on ∂M. Then ∂M is diffeomorphic to a circle and the critical points can be labeled, in cyclic order, $\mathbf{p}_1, \mathbf{p}_1', \mathbf{p}_2, \mathbf{p}_2', \ldots, \mathbf{p}_k, \mathbf{p}_k'$ for some $k \in \mathbb{N}$ where S has maxima at the $\mathbf{p}_i$ and minima at the $\mathbf{p}_i'$. This fixes an orientation of ∂M. Recall the considerations in the proof of Proposition 4.2 in the case of surfaces: The unstable manifold $M^u_{\mathbf{p}}$ is two-dimensional for each maximum $\mathbf{p} = \mathbf{p}_i$, and it projects diffeomorphically to M in a neighborhood of $\overline{\mathbf{p}}$. It is one-dimensional for each minimum $\mathbf{p} = \mathbf{p}_i'$. Then for each i the integral curves of $\mathbf{V}$ starting at $\overline{\mathbf{p}}_i$, considered as sets, foliate $M^u_{\mathbf{p}_i} \setminus \overline{\mathbf{p}}_i$. They may be parametrized by the points of a small closed half circle T_i around $\overline{\mathbf{p}}_i$ in $M^u_{\mathbf{p}_i}$ transversal to $\mathbf{V}$. Identify T_i with the closed interval $[i-1, i]$, so that the parametrization preserves orientation and the boundary points $i-1, i$ correspond to curves in the boundary. In addition $M^u_{\mathbf{p}_i'} \setminus \overline{\mathbf{p}}_i'$ consists of a single interior integral curve, which we label by i. In this way every interior integral curve starting at a boundary point is labeled by a unique number in $(0, k]$. Identifying $\mathbf{S}^1$ with $\mathbb{R}/(k\mathbb{Z})$ we obtain a parametrization of the set of geodesics starting at the boundary by $\mathbf{S}^1$.

Definition 6.1. *The exponential map is defined by the parametrization introduced above, together with arc length parametrization along each geodesic starting at $\tau = 0$.*

Note that the exponential map preserves cyclic ordering, with respect to the chosen orientation of ∂M and the standard orientation of $\mathbf{S}^1$, in the following sense. If $a, b, c \in \mathbf{S}^1$ are pairwise different and lie in this order on $\mathbf{S}^1$ then there is $\tau_0 > 0$ so that the geodesics $\gamma_a, \gamma_b, \gamma_c$ do not intersect and lie in this order. Here, τ_0 may depend on a, b, c, and for certain cusp metrics will in fact not be bounded away from zero, which leads to discontinuity of the exponential map, see Remark 6.5.

Remark 6.2. *The domain of the parametrization $q \mapsto \gamma_q$ may be described somewhat more naturally with the help of the following construction: For a vector field* $\mathbf{V}$ *on a smooth manifold* Σ *with an unstable critical point* $\mathbf{p}$ *there is a natural notion of blowing-up of* $\mathbf{p}$ *in* Σ *with respect to* $\mathbf{V}$. *This is a smooth manifold with boundary, denoted* $[\Sigma, \mathbf{p}]_{\mathbf{V}}$, *together with a blowing-down map* $\beta_{\mathbf{p}} : [\Sigma, \mathbf{p}]_{\mathbf{V}} \to \Sigma$. *The boundary (front face) parametrizes the integral curves of* $\mathbf{V}$ *starting at* $\mathbf{p}$. *It is diffeomorphic to a sphere and may be thought of as small sphere around* $\mathbf{p}$ *transversal to* $\mathbf{V}$. *The map* $\beta_{\mathbf{p}}$ *may not be smooth, but* $\mathbf{V}$ *lifts to a smooth vector field on*

$[\Sigma, \mathbf{p}]_{\mathbf{V}}$, *given by* $s\partial_s$ *near the boundary for a suitable boundary defining function* s. *The construction generalizes to the case where* Σ *is a manifold with boundary,* $\mathbf{p} \in \partial\Sigma$, *and* $\mathbf{V}$ *is tangent to* $\partial\Sigma$, *and then yields a manifold with corners. See* [HMV, Section 2] *for details.*

We apply this to the surface $M^u_{\mathbf{p}_i}$ *for each* i *and obtain* $[M^u_{\mathbf{p}_i}, \overline{\mathbf{p}}_i]_{\mathbf{V}}$. *Then the transversal* T_i *is naturally the front face* $\beta_{\mathbf{p}}^{-1}(\overline{\mathbf{p}}_i)$ *of the blowing-up. The various* T_i *are now glued at their endpoints, which are also identified with the points* $\overline{\mathbf{p}}'_i$, *and this is the first factor* $\mathbf{S}^1$ *of the domain of* $\exp_{\partial M}$.

Proof of Theorem 1.3. We constructed the exponential map above. We need to show that it is surjective. For this we prove that the union of the unstable manifolds $M^u_{\mathbf{p}}$ over all critical points projects *onto* a neighborhood of ∂M under the canonical projection $\pi : {}^3T^*M \to M$.

First, observe that each unstable manifold $M^u_{\mathbf{p}}$ intersects the boundary $\{r = 0\}$ transversally near $\overline{\mathbf{p}}$ since this is true for the linear part of $\mathbf{V}$, and since both $M^u_{\mathbf{p}}$ and $\{r = 0\}$ are invariant under the flow, the intersection is transversal everywhere. The intersection $M^u_{\mathbf{p}} \cap \{r = 0\}$ is the unstable manifold $\Gamma^u_{\mathbf{p}}$ of $\overline{\mathbf{p}}$ of $\mathbf{V}$ restricted to the boundary.

The projection of the unstable manifolds $M^u_{\mathbf{p}'_i}$ are (closures of) single geodesics starting at $\mathbf{p}'_i$, which divide a neighborhood of ∂M into connected components U_i, each containing one maximum $\mathbf{p}_i$. It suffices to show that $\pi(M^u_{\mathbf{p}_i})$ contains U_i for each i.

For this, first consider the intersections with the boundary. Fix i and write $\mathbf{p}_- = \mathbf{p}_i$ and $\mathbf{p}_{\min} = \mathbf{p}'_i$. By the discussion before Lemma 5.3, the part $\Gamma_{\mathbf{p}_-}$ of $\Gamma^u_{\mathbf{p}_-}$ leaving $\mathbf{p}_-$ in the positive φ direction projects onto the (open) interval from $\mathbf{p}_-$ to $\mathbf{p}_{\min}$. A similar statement holds for the interval from $\mathbf{p}'_{i-1}$ to $\mathbf{p}_-$. Therefore, $M^u_{\mathbf{p}_i} \cap \{r = 0\}$ projects onto $U_i \cap \{r = 0\}$. Since $M^u_{\mathbf{p}_i}$ intersects the boundary transversally, it projects onto some neighborhood of the open interval from $\mathbf{p}'_{i-1}$ to $\mathbf{p}'_i$. We need to show that this neighborhood cannot shrink to zero width when one approaches $\mathbf{p}'_{i-1}$ or $\mathbf{p}'_i$. We will prove that U_i intersected with a neighborhood of $\mathbf{p}'_i$ is contained in $\pi(M^u_{\mathbf{p}_i})$; the argument at $\mathbf{p}'_{i-1}$ is then analogous.

Recall the dichotomy (27). If $T < \infty$ here then $\pi(\Gamma_{\mathbf{p}_-})$ contains $\mathbf{p}'_i$, so we are done. On the other hand, if $T = \infty$ then $\Gamma_{\mathbf{p}_-}(t) \to \overline{\mathbf{p}}_{\min}$ as $t \to \infty$, and the eigenvalues $\lambda_2(\mathbf{p}_{\min})$, $\lambda_3(\mathbf{p}_{\min})$ are real. In this case, the behavior of $M^u_{\mathbf{p}_-}$ near $\overline{\mathbf{p}}_{\min}$ may be understood by linearizing $\mathbf{V}$ near $\overline{\mathbf{p}}_{\min}$. We use Lemma 6.3 below, applied as explained after its statement. For later purposes the lemma is stated more strongly than needed here. Here we only need the consequence that, in a neighborhood of $\overline{\mathbf{p}}_{\min}$, the curve $M^u_{\mathbf{p}_{\min}}$ is contained in the closure of $M^u_{\mathbf{p}_-}$, so the projection of $M^u_{\mathbf{p}_{\min}}$ is contained in the closure of U_i. This completes the proof of Theorem 1.3.

Lemma 6.3. *Let* $\lambda, \mu < 0$ *and let* V *be the linear vector field* $x_1\partial_{x_1} + \lambda x_2\partial_{x_2} + \mu x_3\partial_{x_3}$ *on* $\mathbb{R}^3$. *Let* $m \geq 1$. *Let* Σ *be an invariant* C^m *surface with boundary* $\Gamma = \Sigma \cap \{x_1 = 0\}$, *and assume this intersection is transversal and* Γ *is a single trajectory of* V. *Denote by* X_1 *the non-negative* x_1*-axis.*

Then a neighborhood Σ' *of zero in* $\Sigma \cup X_1$ *is a surface with corner. More precisely, let* $\tau = -\frac{1}{\lambda}$ *and* $\rho = \frac{\mu}{\lambda}$, *so* $\tau, \rho > 0$. *Assume* $\Gamma \subset \{x_2 > 0\}$. *Then* Σ' *is a graph*

$$\Sigma' = \{(x_1, x_2, x_3) : \ x_3 = A(x_1 x_2^\tau) x_2^\rho, \ (x_1, x_2) \in U'\} \tag{31}$$

where A is a C^m function on $[0,\varepsilon)$ for some $\varepsilon > 0$, and U' is a neighborhood of the origin in the quarter plane $\mathbb{R}_+ \times \mathbb{R}_+$.

The lemma is applied as follows, in the context of the discussion before Lemma 5.3: Let $\mathbf{p}_{\min}$ be a minimum of S, and suppose the second alternative in (27) holds. Then $\Gamma_{\mathbf{p}_-}(t)$ approaches $\overline{\mathbf{p}}_{\min}$ as $t \to \infty$. By Proposition 4.6 we may linearize $\mathbf{V}$ near $\overline{\mathbf{p}}_{\min}$ by a C^m-diffeomorphism where $m \in \{1,2\}$. Thus, we introduce local coordinates x_1, x_2, x_3 so that $\overline{\mathbf{p}}_{\min}$ is the origin and $\mathbf{V}$ is given by $x_1\partial_{x_1} + \lambda x_2 \partial_{x_2} + \mu x_3 \partial_{x_3}$ on a neighborhood U where $\{\lambda, \mu\} = \{\lambda_2(\mathbf{p}_{\min}), \lambda_3(\mathbf{p}_{\min})\}$ and the boundary is given by $\{x_1 = 0\}$. We take $\Gamma = \Gamma_{\mathbf{p}_-}$ and $\Sigma = M^u_{\mathbf{p}_-}$. Clearly $M^u_{\mathbf{p}_{\min}} = X_1$. Since Γ is a trajectory of $\mathbf{V}$ approaching the origin, it must have a tangent vector there, and we may assume that the coordinates are chosen so that this vector is ∂_{x_2}. Then $\Gamma \subset \{x_2 > 0\}$ near the origin, so the assumptions of the lemma are satisfied.

Proof of Lemma 6.3. Choose a point p of Γ. Since the intersection of Σ with $\{x_1 = 0\}$ is transversal we may choose a C^m-curve $\omega(s) = (s, \omega_2(s), \omega_3(s))$, $s \in I = [0,\varepsilon)$ contained in Σ, with $\omega(0) = p$, so $\omega_2(0) > 0$. Also, since $x_1 x_2^\tau$ is constant along any integral curve of $\mathbf{V}$, only trajectories passing through this curve will contribute to Σ' if this is chosen sufficiently small.

For $s \in I$ let $\mathbb{R}_+ \to \mathbb{R}^3$, $t \mapsto \Gamma_s(t)$ be the forward integral curve of $\mathbf{V}$ starting at $\omega(s)$, so $\Gamma_s(0) = \omega(s)$ and $\Gamma_0 = \Gamma$, up to time shift.

For each s, the quantities $x_1 x_2^\tau$ and $x_3 x_2^{-\rho}$ are constant along Γ_s, that is, for points (x_1, x_2, x_3) in the image of Γ_s. Evaluating at $t = 0$ shows that, along Γ_s,

$$x_1 x_2^\tau = s\omega_2(s)^\tau, \quad x_3 x_2^{-\rho} = \omega_3(s)\omega_2(s)^{-\rho}. \tag{32}$$

Now $\omega_2(0) > 0$, hence the equation $\sigma = s\omega_2(s)^\tau$ can be solved C^m-smoothly for $s = s(\sigma)$, for σ in a half neighborhood J of zero, and the function $A : J \to \mathbb{R}$, $\sigma \mapsto \omega_3(s(\sigma))\omega_2(s(\sigma))^{-\rho}$ is C^m. Then we have along Γ_s, with this σ,

$$x_3 = A(\sigma) x_2^\rho = A(x_1 x_2^\tau) x_2^\rho.$$

The claim follows. ◻

Proof of Theorem 1.4. The heart of the proof is that the various unstable manifolds fit together nicely. So we first prove the following proposition.

Proposition 6.4. *Assume the setting and the conditions of Theorem 1.4. Let M^u be the union of the unstable manifolds $M^u_{\mathbf{p}}$, where $\mathbf{p}$ ranges over the critical points of S. Then M^u is a C^1 manifold, and there is $r_0 > 0$ so that the intersection of M^u with $\{r < r_0\}$ projects diffeomorphically to $M \cap \{r < r_0\}$ under the projection $\pi : {}^3T^*M \to M$.*

See Figure 4.

The proof actually gives more regularity than C^1: The manifold M^u is C^∞ except at points on the curves $M^u_{\mathbf{p}}$ where $\mathbf{p}$ is a minimum of S, and here the regularity is $C^{\alpha(\mathbf{p})}$ where

$$\alpha(\mathbf{p}) = \begin{cases} 1 & \text{if } \lambda_2 \in \mathbb{Q} \\ \min\{2, \frac{\lambda_3}{\lambda_2}\} & \text{if } \lambda_2 \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \tag{33}$$

with $\lambda_2 = \lambda_2(\mathbf{p})$, $\lambda_3 = \lambda_3(\mathbf{p})$.

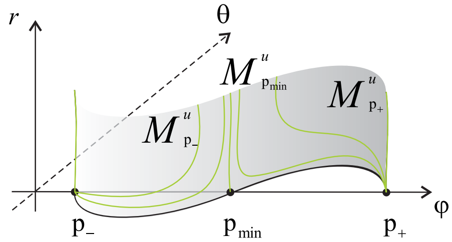

FIGURE 4. The dynamics near the boundary.

Proof of Proposition 6.4. The main task is to prove that the different $M^u_{\mathbf{p}}$ fit together to form a C^1 manifold, the projection statement will then be seen to be an easy consequence.

Since integral curves do not intersect, the manifolds $M^u_{\mathbf{p}}$ do not intersect for different $\mathbf{p}$. Since M^u is invariant under the flow, it is enough to prove the regularity near $r = 0$.

First consider the boundaries $\Gamma^u_{\mathbf{p}} = \partial M^u_{\mathbf{p}} = M^u_{\mathbf{p}} \cap \{r = 0\}$ and their union $\Gamma^u = \bigcup_{\mathbf{p}} \Gamma^u_{\mathbf{p}}$. These are the unstable manifolds of $\mathbf{V}$ restricted to the boundary $T^*\partial M$. By assumption $S_{\varphi\varphi} < \frac{9}{4}$ everywhere, so we may apply Proposition 5.4 and conclude that Γ is the graph of a C^1 function $\Gamma = \{(\varphi, h(\varphi)) : \varphi \in \partial M\}$. Thus, both the regularity and the projection statement of Proposition 6.4 are true for the boundary of M^u.

We now prove that M^u is a C^1 manifold in a neighborhood of $r = 0$, with boundary Γ^u. Since $M^u_{\mathbf{p}}$ is a smooth surface for each maximum $\mathbf{p}$ of S, we only need to prove the regularity in a small neighborhood U of $\overline{\mathbf{p}}_{\min}$ where $\mathbf{p}_{\min}$ is a minimum of S. Thus, let $\mathbf{p}_{\min}$ be a minimum of S and let $\mathbf{p}_-$, $\mathbf{p}_+$ be the maxima of S closest to $\mathbf{p}_{\min}$, so that the corresponding parameter values satisfy $\varphi_- < \varphi_{\min} < \varphi_+$. Then the set $M^u \cap U$ is the union of $M^u_{\mathbf{p}_-} \cap U$, $M^u_{\mathbf{p}_{\min}} \cap U$ and $M^u_{\mathbf{p}_+} \cap U$. We now apply Lemma 6.3 as explained after its statement, first to $\Sigma_- = M^u_{\mathbf{p}_-} \cap U$ and then to $\Sigma_+ = M^u_{\mathbf{p}_+} \cap U$. By the last statement of Proposition 5.4, $\Gamma_{\mathbf{p}_\pm}$ have tangents $\pm\nu_2(\mathbf{p}_{\min})$ at $\overline{\mathbf{p}}_{\min}$, so we take $\lambda = \lambda_2(\mathbf{p}_{\min})$, $\mu = \lambda_3(\mathbf{p}_{\min})$ in Lemma 6.3, and this implies $\rho > 1$ there. The lemma shows that $\Sigma_\pm$ are graphs as in (31), with possibly different functions $A_\pm$.

The more precise regularity statement (33) follows from the regularity in Lemma 6.3 together with the regularity of the linearization, Proposition 4.6.

Finally we prove that M^u projects diffeomorphically to M near $r = 0$. For any boundary point $\overline{\mathbf{q}} \in \Gamma^u = \partial M^u$, the tangent space $T_{\overline{\mathbf{q}}} M^u$ is transversal to $\{r = 0\}$, since this is already true for each $M^u_{\mathbf{p}}$ separately. In addition, $T_{\overline{\mathbf{q}}} M^u \supset T_{\overline{\mathbf{q}}} \Gamma^u$, so $T_{\overline{\mathbf{q}}} M^u \cap T_{\overline{\mathbf{q}}}\{r = 0\}$ contains a vector with nonzero ∂_φ-component. These two facts combine to show that, for each $\overline{\mathbf{q}} \in \partial M^u$, the tangent space $T_{\overline{\mathbf{q}}} M^u$ projects isomorphically to $T_{\mathbf{q}} M$ under $d\pi$, where $\pi : {}^3T^*M \to M$ is the projection, and this implies the claim. □

We now prove Theorem 1.4. By Theorem 1.1 every geodesic starting at ∂M must do so at a critical point $\mathbf{p}$ of S, so the corresponding integral curve of $\mathbf{V}$ starts at a singular point $\overline{\mathbf{p}}$ of $\mathbf{V}$. This integral curve is then contained in the

unstable manifold $M^u_{\mathbf{p}}$ of $\overline{\mathbf{p}}$. Conversely, $M^u_{\mathbf{p}} \cap \{r > 0\}$ is the union of such integral curves. Therefore, Proposition 6.4 implies that $\exp_{\partial M}$ is bijective.

It remains to prove the continuity of $\exp_{\partial M}$ and of its inverse. We continue to work on M^u. Some care needs to be taken since the vector field $\mathbf{W}$ blows-up at the boundary. But the main point is that only its ∂_φ component blows-up, while its ∂_r component is $\xi + O(r^2) = 1 + O(r^2)$. Recall that the domain of $\exp_{\partial M}$ is the set of (q, τ) where $\tau \in (0, \tau_0)$ and q is a point on a transversal T_i near $\overline{\mathbf{p}}_i$ for some i, where the transversals are glued at their endpoints. The continuity of $\exp_{\partial M}$ at points (q_0, τ) where q_0 is not a boundary point of a transversal T_i is clear, so we assume that q_0 is a boundary point of T_i, say the 'right' boundary point, which labels the geodesic starting at the next minimum $\mathbf{p}'_i$. The idea is this: As sets, the geodesics γ_q, with q in the interior of T_i approaching q_0, converge to the union of the boundary trajectory $\Gamma_{\mathbf{p}_i}$ and of $M^u_{\mathbf{p}'_i}$. Since $r' = \frac{dr}{d\tau}$ is approximately one, the part of γ_q near $\Gamma_{\mathbf{p}_i}$ is traversed in a very short time. Then the part near $M^u_{\mathbf{p}'_i}$ must, including its time parametrization, be close to γ_{q_0}.

More precisely, fix (q_0, τ) and a neighborhood U of $\gamma_{q_0}(\tau)$. Denote by r, r_0 the r-components of integral curves γ, γ_{q_0} of $\mathbf{W}$. Then there are $\delta > 0$, a neighborhood U' of $M^u_{\mathbf{p}'_i}$ and a neighborhood V' of τ so that for all $0 < \tau_1 < \tau$ and all integral curves γ of $\mathbf{W}$ lying in U' for the time interval (τ_1, τ) and satisfying $|r(\tau_1) - r_0(\tau_1)| < \delta$ we have that $\gamma(\tau') \in U$ for all $\tau' \in V'$. This is because $r' = F(r, \varphi, \theta)$ where F is C^1 and the values of φ, θ for γ and γ_0 will be close together at all times in (τ_1, τ) if U' is chosen sufficiently small.

Next, for the $\delta > 0$ and neighborhood U' of $M^u_{\mathbf{p}'_i}$ obtained above there is a neighborhood V of q_0 in T_i so that for all $q \in V$ the curve γ_q first runs inside $\{r < \delta/2\}$ and then inside U'. Now $\frac{dr}{d\tau} = 1 + O(r^2)$ implies that the travel time $\tau_1(q)$ for the first part is at most on the order of $\delta/2$. Then the r-components r_q and r_0 of γ_q, γ_{q_0} satisfy $|r_q(\tau_1(q)) - r_0(\tau_1(q))| < \delta$, so the first part of the argument can be applied with $\gamma = \gamma_q$ and $\tau_1 = \tau_1(q)$. Summarizing, given any neighborhood U of $\gamma_0(\tau)$ we have found neighborhoods V of q_0 and V' of τ so that $\gamma_q(\tau') \in U$ for all $q \in V$, $\tau' \in V'$. This proves the continuity of $\exp_{\partial M}$, from the left with respect to q. Continuity from the right follows by the same argument applied to the left endpoint of T_{i+1}. The continuity of the inverse is proved in a similar way, but easier. For example, the second component, τ, of $\exp^{-1}_{\partial M}$ is simply the distance to the boundary, and its continuity follows from the triangle inequality.

Remark 6.5. *The proof also shows why the exponential map may be discontinuous in general: Suppose $k \geq 2$, i.e. the function S has at least two maxima and at least two minima. Suppose $\Gamma_{\mathbf{p}_1}$ does not approach $\mathbf{p}'_1$ but rather continues in the upper half plane $\theta > 0$ and then approaches $\mathbf{p}'_2$. Examples of functions S yielding this boundary dynamics can easily be constructed. Let q_1, q_2 be the labels of the geodesics leaving $\mathbf{p}'_1$, $\mathbf{p}'_2$, respectively. Then, for any fixed $\tau > 0$, the point $\gamma_q(\tau)$ will approach $\gamma_{q_2}(\tau)$ rather than $\gamma_{q_1}(\tau)$ as $q \to q_1$ from the left, so $\exp_{\partial M}$ is discontinuous at (q_1, τ) for any $\tau > 0$. Also, it is easy to see that this discontinuity cannot be removed by a simple reordering (i.e. by a different gluing prescription), which in any case would be unnatural since it would break up the order preserving property of* $\exp_{\partial M}$.

Remark 6.6. *The proof of Theorem 1.4 gives more than is stated in the theorem. For example, the restriction of the smooth vector field* $\mathbf{V}$ *on* ${}^3T^*M$ *to the* C^1*-submanifold* M^u *may be projected to* M*. This yields a* C^1 *vector field* $\widetilde{V}$ *on* M *(which is* C^∞ *away from the finitely many geodesics leaving minima of* S*) whose integral curves are the geodesics starting at* ∂M*, as sets. The vector field* $\widetilde{\mathbf{W}} = r^{-1}\widetilde{V}$ *on* $\mathring{M}$ *has, by definition, unit length everywhere, and it is the gradient field of the distance function to the boundary,* $\widetilde{\mathbf{W}} = \nabla\mathrm{dist}$*. Note also that* $r' = \xi + O(r^2) = 1 + O(r^2)$ *yields that* $r = g\mathrm{dist}$ *for a non-vanishing function* g *equal to one at the boundary, and therefore we get* $\widetilde{V} = g\nabla\mathrm{dist}^2$*. So the gradient of* dist^2 *has a certain amount of smoothness.*

Remark 6.7. *The blowing-up procedure explained in Remark 6.2 can be used to describe the boundary behavior of* $\exp_{\partial M}$*, in the setting of Theorem 1.4. Let* $\widetilde{M}$ *be the space obtained by blowing-up all maxima of* S *in* M *with respect to the geodesic vector field* $\widetilde{V}$ *(see Remarks 6.2 and 6.6), with blowing-down map* $\beta : \widetilde{M} \to M$*. Also, let* M' *be the space obtained from* $\widetilde{M}$ *by collapsing all intervals between adjacent front faces to points; these intervals correspond to intervals on* ∂M *from one maximum to the next. As explained in Remark 6.2 the* $\mathbf{S}^1$ *part in the domain of* $\exp_{\partial M}$ *may be naturally understood as the union of all the front faces of* β*, glued at their ends in cyclic order, hence with the boundary of* M'*. Then* $\exp_{\partial M}$ *extends to a homeomorphism from* $\mathbf{S}^1 \times [0, \tau_0)$ *to a neighborhood of the boundary in* M'*.*

Note that M*,* $\widetilde{M}$ *and* M' *are different ways of adding a 'boundary' to* $\mathring{M}$*. The correspondence of* M' *and* M *is analogous to birational maps in algebraic geometry. Both are obtained as blowings-down (contracting certain sides to points) of the manifold with corners* $\widetilde{M}$*:*

$$M' \longleftarrow \widetilde{M} \longrightarrow M$$

Proof of Theorem 1.5. Let $\mathbf{p}_{\min}$ be a point where S has a local minimum and where $S_{\varphi\varphi} > \frac{9}{4}$. Then the eigenvalues $\lambda_2(\mathbf{p}_{\min})$, $\lambda_3(\mathbf{p}_{\min})$ are non-real with real part $-\frac{3}{2}$, so nearby trajectories of $\mathbf{V}$ on $T^*\partial M$ spiral towards $\overline{\mathbf{p}}_{\min}$.

Consider the maxima $\mathbf{p}_-$, $\mathbf{p}_+$ of S closest to $\mathbf{p}_{\min}$, where $\varphi_- < \varphi_{\min} < \varphi_+$ for the corresponding parameters. W.l.o.g. we may assume $S(\varphi_-) \leq S(\varphi_+)$. Then we are in the situation discussed before Lemma 5.3 (where we put $\varphi_- = 0$ for simplicity), more precisely in the first case of (27). We claim that the trajectory $\Gamma_{\mathbf{p}_-}$ considered there approaches $\overline{\mathbf{p}}_{\min}$ as $t \to \infty$. To show this, recall from Proposition 5.1 that the function $S(\varphi) + \theta^2/2$ is strictly decreasing along $\Gamma_{\mathbf{p}_-}$. At $t = -\infty$ this function has the value $S(\varphi_-)$, and on the line $\varphi = \varphi_+$ its values are at least $S(\varphi_+) \geq S(\varphi_-)$. Hence, $\Gamma_{\mathbf{p}_-}$ can not cross or approach this line, and neither the line $\varphi = \varphi_-$. So the only possible limit point of $\Gamma_{\mathbf{p}_-}$ for $t \to \infty$ is $\overline{\mathbf{p}}_{\min}$, which was to be shown.

Now note that the spiraling of $\Gamma_{\mathbf{p}_-}$ around $\overline{\mathbf{p}}_{\min}$ as $t \to \infty$ implies that the projection of $\Gamma_{\mathbf{p}_-}$ to the φ-axis oscillates around $\varphi_{\min}$ infinitely often.

Now consider the unstable manifold $M^u_{\mathbf{p}_-}$. The image of $\Gamma_{\mathbf{p}_-}$ is an open subset of $\partial M^u_{\mathbf{p}_-}$. By continuity of $\mathbf{V}$, for any $T > 0$ and $\varepsilon > 0$ there are interior trajectories of $\mathbf{V}$ contained in $M^u_{\mathbf{p}_-}$ that follow the boundary trajectory $\Gamma_{\mathbf{p}_-}$ within an error of ε on the time interval $[0, T]$. In particular, for any $\varepsilon > 0$ there is a trajectory γ starting at $\overline{\mathbf{p}}_-$ whose projection to M intersects the projection of the trajectory $M^u_{\mathbf{p}_{\min}}$ at a point with $r < \varepsilon$. This proves the theorem.

7. Examples

We consider surfaces $X \subset \mathbb{R}^3$ given as in (2) with $\widetilde{X} = \partial\widetilde{X} \times \mathbb{R}_+$ a cylinder, for different boundary curves $\partial\widetilde{X} \subset \mathbb{R}^2$. We use the Euclidean metric on $\mathbb{R}^3$. Write coordinates on $\mathbb{R}^2$ as $u = (v, w)$. Recall that $S(v, w) = v^2 + w^2$, see Remark 2.4.

(1) If $\partial\widetilde{X}$ is a circle centered at the origin then S is constant, the geodesics of X starting at the origin foliate the surface. This is obvious by rotational symmetry.

(2) If $\partial\widetilde{X}$ is an ellipse centered at the origin,
$$\frac{v^2}{c^2} + \frac{w^2}{b^2} = 1, \text{ with } c > b > 0.$$
The function S has two maxima at $(\pm c, 0)$ and two minima at $(0, \pm b)$. The boundary $\partial\widetilde{X}$ is simply tangent to the circles centered at the origin passing through these points. Since $\partial\widetilde{X}$ bounds a strictly convex set containing the origin, Theorem 1.4 is applicable, see Proposition 5.5. There is one geodesic starting from each of the points $(0, \pm b)$, all the others start at $(\pm c, 0)$.

(3) We now consider a circle not having the origin in its interior,
$$(v - c)^2 + w^2 = 1, \quad c > 1.$$
The function S has a minimum at $\mathbf{p}_{\min} = (c - 1, 0)$ and a maximum at $\mathbf{p}_{\max} = (c + 1, 0)$. These are non-degenerate, so we have simple tangency again. In the arc length parametrization $v(\varphi) = c + \cos\varphi$, $w(\varphi) = \sin\varphi$, we find $S(\varphi) = c^2 + 1 + 2c\cos\varphi$, so $S_{\varphi\varphi}(\varphi) = -2c\cos\varphi$. At the minimum $\varphi = \pi$ we see that $S_{\varphi\varphi} = 2c$, which is also the maximal value of $S_{\varphi\varphi}$. We obtain the following picture:

There is a single geodesic $\gamma_{\min}$ starting at $\mathbf{p}_{\min}$, all others start from $\mathbf{p}_{\max}$. Their behavior depends on c:
If $1 < c < 9/8$ then Theorem 1.4 is applicable.
If $c > 9/8$ then we are in the situation of Theorem 1.5. Geodesics starting at $\mathbf{p}_{\max}$ almost tangentially to $\partial\widetilde{X}$ will approach $\mathbf{p}_{\min}$, then oscillate around $\gamma_{\min}$ many times before they escape far away from the singularity.

Our first and simplest example above might suggest that Theorem 1.2 is not so interesting. However, the point of the theorem is that all that matters for the conclusion is the constancy of S *on the boundary.* Hence, any surface $\widetilde{X}$ with the same boundary will have a foliation by geodesics near the singularity. As another example, consider a surface $\widetilde{X}$ embedded in $\mathbb{R}^n$ with $n > 3$, with $\partial\widetilde{X}$ any curve in a sphere $R \cdot \mathbf{S}^{n-2}$. Then there is no rotational symmetry, not even for $\partial\widetilde{X}$, but Theorem 1.2 still gives a foliation by geodesics.

8. Higher order cusps

Everything carries over to higher order cusps, with minor numerical changes. Let $k \geq 2$. A *cusp singularity of order* k is one which can be resolved by an order k blowing-up $(u, z) \mapsto (z^k u, z)$, or alternatively by a sequence of k standard blowings-up, or alternatively by one standard blowing-up followed by an order $(k - 1)$ blowing-up. Using this last characterization the proof of Proposition 2.3 carries over, mutatis mutandis, and shows that the resolution of a cusp singularity

of order k yields a *k-cusp metric* on the manifold with boundary $M = \widetilde{X}$, which is of the form

$$\mathbf{g}_{|U} = (1 - k(k-1)r^{2k-2}S + O(r^{2k-1}))\, dr^2 + r^{2k}h$$

where S and h are as before. The factor $k(k-1)$ is put in so that in the case of $M = \widetilde{X}$ the function S is determined by (11) as before.

The correct rescaling of the cotangent variable is

$$\theta = \frac{\eta}{r^{2k-1}},$$

and then the energy is given by $\frac{1}{2}\xi^2 + r^{2k-2}(G_0 + \widetilde{G})$ where

$$G_0 = \tfrac{1}{2}k(k-1)S\xi^2 + \tfrac{1}{2}C^{-1}\theta^2.$$

The rescaled geodesic vector field is $\mathbf{V} = r\mathbf{W}$ and near the boundary yields the equations, generalizing (18),

$$\begin{aligned} \dot{r} &= r\xi + \dots & \dot{\xi} &= k(1-\xi^2) + \dots \\ \dot{\varphi} &= (G_0)_\theta + \dots & \dot{\theta} &= -(G_0)_\varphi - (2k-1)\xi\theta + \dots \end{aligned}$$

where the dots indicate higher order terms in r. This has linear part as in (19) with S replaced by $\frac{1}{2}k(k-1)S$ and 3Id by $(2k-1)$Id. Its eigenvalues (in the case of surfaces, say) are 1 and

$$\frac{-(2k-1) \pm \sqrt{(2k-1)^2 - 2k(k-1)a}}{2}$$

where $a = S_{\varphi\varphi}$ in arc length parametrization. Therefore, the value of a which separates real from non-real eigenvalues is

$$a_k = \frac{(2k-1)^2}{2k(k-1)} = 2 + \frac{1}{2k(k-1)}.$$

The analysis of the boundary dynamics carries over literally if one replaces S by $\frac{1}{2}k(k-1)S$ and 3θ by $(2k-1)\theta$, and $\frac{9}{4}$ by a_k. The barrier function should be taken as $-\frac{k(k-1)}{(2k-1)}S_\varphi$.

Since $(2k-1)\lambda_1(\mathbf{p}) + \lambda_2(\mathbf{p}) + \lambda_3(\mathbf{p}) = 0$, there are resonance relations among $\lambda_1(\mathbf{p}), \lambda_2(\mathbf{p}), \lambda_3(\mathbf{p})$. The value that the function $S_{\varphi\varphi}$ must avoid at a local minimum in the statement of Theorem 1.4 for k-cusp is still 2, where φ is an arc-length parameterization of the boundary curve. Note that $\lambda_2(\mathbf{p}) = 1-k$ iff $a(\mathbf{p}) = 2$. The analogue of Proposition 4.6 holds, but when $\lambda_2 \notin \mathbb{Q}$, we can locally linearize $\mathbf{V}$ by means of a C^{2k-2} diffeomorphism. If $\lambda_2 \in \mathbb{Q}\setminus\{1-k\}$, we can only locally linearize $\mathbf{V}$ by means of a C^1 diffeomorphism.

Now the proofs of all theorems carry over almost literally, with $\frac{9}{4}$ replaced by a_k. Note that a_k strictly decreases to 2 as $k \to \infty$, so the condition on S for having a foliation by geodesics near the singularity becomes (slightly) stronger the larger k is.

Finally, Theorem 1.4 applies to resolutions of cusp singularities of any order $k \geq 2$ satisfying the convexity hypothesis of Proposition 5.5 since $a_k > 2$ for all k. This is precisely what is needed, see the statement of Proposition 5.5.

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